

# Analytic Theory II: Extensive-Form Games of Complete Information (Review)

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February 18, 2020

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# 1 The Extensive Form

Any situation that we wish to represent formally would have some basic elements that will be part of its description. Most often, we begin with a verbal description (that may be quite vague at times), and then distill each element from it. Let's start with a simple card game borrowed from Roger Myerson that we saw already. To refresh our memory, here's the game again.

EXAMPLE 1. (MYERSON'S CARD GAME.) There are two players, labeled "player 1" and "player 2."<sup>1</sup> At the beginning of this game, each player puts a dollar in a pot. Next, player 1 draws a card from a shuffled deck of cards in which half the cards are red and half are black. Player 1 looks at his card privately and decides whether to raise or fold. If player 1 folds, then he shows his card to player 2 and the game ends; player 1 takes the money in the pot if the card is red, but player 2 takes the money if the card is black. If player 1 raises, then he adds another dollar to the pot and player 2 must decide whether meet or pass. If she passes, the game ends and player 1 takes all the money in the pot. If she meets, she puts another dollar in the pot, and then player 1 shows his card to player 2 and the game ends; if the card is red, player 1 takes all the money in the pot, but if it is black, player 2 takes all the money.

The essential elements of a game are:

1. **players:** The individuals who make decisions.
2. **rules of the game:** Who moves when? What can they do?
3. **outcomes:** What do the various combinations of actions produce?
4. **payoffs:** What are the players' preferences over the outcomes?
5. **information:** What do players know when they make decisions?
6. **chance:** Probability distribution over chance events, if any.

A **player** is a decision-maker who is participant in the game and whose goal is to choose the actions that produce his most preferred outcomes or lotteries over outcomes. We assume that *players are rational*: their preference orderings are complete and transitive. We model uncertainty over outcomes with lotteries, like we've done before. This means that preferences can be described with utility functions and rational players choose actions that maximize their expected utilities (that's why we need the vNM theorem).

Let  $\mathcal{I} = \{1, 2, \dots\}$  denote the set of players indexed by  $i$ . That is,  $i \in \mathcal{I}$  is a generic member of this set. In our example,  $\mathcal{I} = \{1, 2\}$ , the two players labeled "player 1" and "player 2."

We represent **chance** events by a *random move of nature*. **Nature**, denoted by  $N$ , is a pseudo-player whose actions are purely mechanical and probabilistic; that is, they determine the probability distribution over the chance events. In our example, Nature "chooses" the color of the card that player 1 randomly draws from deck. Because the number of red cards equals the number of black cards and the deck is shuffled, the probability of the randomly chosen card being red is 0.5. Fig. 1 (p. 3) shows how the random draw by player 1 can be represented as a move by Nature.

Nature "moves" first, and so the **initial node** (or the "root node") of the game, denoted with an empty circle, is the place where the chance event occurs. The two possible "actions" by Nature are *red* and *black*, which we represent with one **branch** each.

Each branch then leads to a **decision node** (denoted with a filled circle), where player 1 gets to make his choice between raising and folding. When player 1 gets to move, he knows the color of the card he has drawn. In our example, player 1 chooses whether to raise or fold under two distinct circumstances,

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<sup>1</sup>We establish the following convention: odd-numbered players are male, and even-numbered players are female. For a generic player, we shall always use the generic male pronoun.

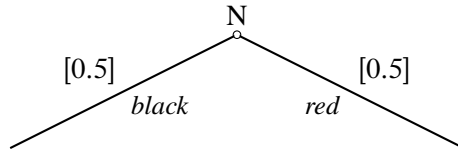


Figure 1: Move by Nature Determines Card Color.

depending on the color of the card. That is, he has one decision to make conditional on the card being black, and another conditional on the card being red. In both cases, the choices are between raising and folding.

We need a way to represent the fact that when player 1 gets to move, he knows the color of the card he is holding. An **information set** for some player  $i$  summarizes what the player knows when he gets to move. Player 1 has two information sets, labeled “b” and “c”. At information set “b”, player 1 knows that the card is black, and at information set “c”, he knows that the card is red. Each of these information sets contains exactly one decision node.

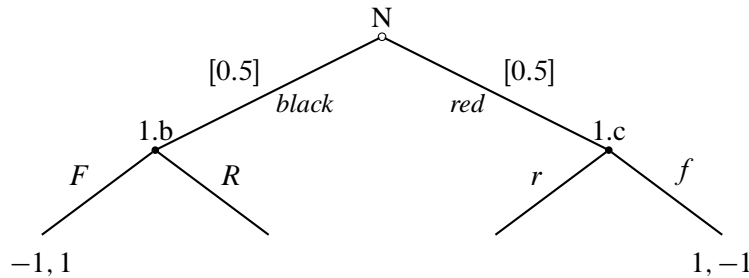


Figure 2: Move by Nature Followed by Choice by Player 1.

For each of his information sets, a player must choose what to do. An **action** (or move) for player  $i$  is a choice, denoted by  $a_i$  that player  $i$  can make at that information set. Let  $A_i = \{a_i\}$  denote the set of choices at an information set. That is, this is the set of actions from which the player must choose. The set of actions may be different depending on the information set. Let  $h$  denote an arbitrary information set (we shall shortly see why this letter is appropriate). Then  $A_i(h)$  is the set of actions available to player  $i$  at information set  $h$ . If the player does not get to move at information set  $h$ , then  $A_i(h) = \emptyset$ .

In our example, player 1 always has the same two actions regardless of the color of the card: He can either raise, denoted by  $R$ , or fold, denoted by  $F$ . Thus,  $A_1(b) = \{F, R\}$  and  $A_1(c) = \{f, r\}$ . We represent the actions available at a decision node with branches emanating from that node, as shown in Fig. 2 (p. 3). I have used upper and lower case letters to denote the actions at the different information sets to emphasize that they are, in fact, different in the sense that although the action is the same it occurs in a different context. That is, even though  $F$  and  $f$  both represent the action “fold,” the first is really “fold on black card” and the second is “fold on red card.”

Information sets that contain only one decision node are called **singletons**. Here, both information sets for player 1 are singletons. Note that we have labeled the two information sets by player 1 with “1.b” and “1.c” respectively. This is intended to convey both that player 1 gets to move and that he knows different things at the different information sets.

A **history** of the game is a sequence of actions taken by the various players at their information sets. The initial history (before the game begins) is denoted by  $h^0 = \emptyset$ . One history of the game is (black), that is, nature having chosen black. Another history is (black,  $F$ ), that is, nature having chosen black, and player 1 having folded.

More generally, we can think of the game as a sequence of stages, where all players simultaneously choose actions from their choice sets  $A_i(h)$  (remember that these choices may be “do nothing” if the player’s action set is empty at  $h$ ). An **action profile** is the set of actions taken by the players at that stage. For example,  $h^0$  is the “history” at the beginning of the game, and  $a^0 = (a_1^0, \dots, a_I^0)$  is the action profile following  $h^0$ . Then  $h^1$  is the history identified with  $a^0$ , and  $A_i(h^1)$  is the set of actions available to player  $i$  there. Continuing iteratively in this manner, we define the history at the end stage  $k$  to be the sequence of actions in the previous stages:

$$h^{k+1} = (a^0, a^1, \dots, a^k).$$

We shall let  $K + 1$  denote the total number of stages in the game, noting that for some games, we may have  $K = +\infty$ . In these cases, the “outcome” of the game is the infinite history  $h^\infty$ . Let  $H = \{h^k\}$  denote the set of all possible histories. Since each  $h^{K+1}$  by definition describes an entire sequence of actions from the beginning of the game to its end, we shall call it a **terminal history**. The set  $\mathcal{Z} = \{h^{K+1}\} \subset H$  of all terminal histories is the same as the set of outcomes when the game is played.

Returning to our example, the history (red,  $f$ ) is terminal because the game ends if player 1 folds. Conversely, the histories (red) and (red,  $r$ ) are not terminal because the game continues. Note that information sets are related to histories because they summarize past play and what players know about it.

For each player  $i$ , we specify a **payoff function**,  $u_i : \mathcal{Z} \rightarrow \mathbb{R}$ . That is, a function that maps the set of terminal histories (or **outcomes**), to real numbers. In other words, we assign numeric payoffs to the outcomes. Of course, this function must represent the preference ordering of the player over the outcomes. Since  $h_1 = (\text{black}, F)$  and  $h_2 = (\text{red}, f)$  are both terminal histories, the player’s (Bernoulli) payoff functions must assign numbers to these outcomes.<sup>2</sup> Let’s assume that utilities are linear in the amount of money received, or  $u(z) = z$ . Then:

$$\begin{aligned} u_1(h_1) &= u_2(h_2) = -1 \\ u_1(h_2) &= u_2(h_1) = 1. \end{aligned}$$

We list these payoffs below the terminal node associated with them. By convention, the order is determined by the order in which players appear in the game tree, top to bottom and left to right. In our example in Fig. 2 (p. 3), the first number is player 1’s payoff and the second number is player 2’s payoff.

If player 1 raises, player 2 gets to make a move. Thus, the  $R$  and  $r$  branches representing raising by player 1 lead to decision nodes for player 2. She can either meet,  $m$ , or pass,  $p$ , and so each decision node will have two branches, labeled  $m$  and  $p$  respectively, as shown in Fig. 3 (p. 5). The payoffs from the resulting terminal histories are specified in the same manner as before.

The crucial difference between the information available to player 1 and the information available to player 2 is that player 2, unlike player 1, does not know the color of player 1’s card although she does observe his action (raising). In other words, when player 2 gets to move, she does not know whether player 1’s card is red or black. The information set, denoted by “0” for player 2 thus includes *both* histories  $h_3 = (\text{black}, R)$  and  $h_4 = (\text{red}, r)$ . Because each of these histories leads to a different decision node for player 2, we enclose them in a box (or connect them in some other way) to demonstrate that they belong to the same information set. We say that both  $h_3$  and  $h_4$  are **consistent** with the information set “0”. The information set represents the fact that when player 2 gets to move, she does not know the color of the card; she only knows what she can see—namely, that player 1 has chosen to raise.<sup>3</sup>

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<sup>2</sup>Bernoulli defined the utility function over wealth, and by convention we use the term *Bernoulli function* to refer to payoff functions defined over the outcomes. Von Neumann and Morgenstern moved away from this and defined the *expected utility function* over lotteries. People sometimes call these *Von Neumann-Morgenstern Utility Function* or, simply, *vNM Utility Function*. Recall that these are subjective in the sense that preferences must be given before these utilities can be derived.

<sup>3</sup>As we shall see when we analyze the game, in equilibrium player 2 may learn about the likelihood of the card’s color by using

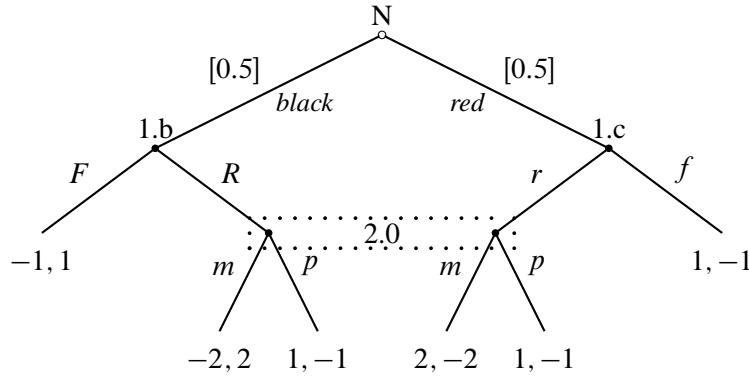


Figure 3: Myerson's Card Game in Extensive Form.

Player 2's information set is not a singleton because it contains two of her decision-nodes. Let  $h(x)$  denote the fact that the information set  $h$  contains node  $x$ . The information set captures the idea that the player who is choosing an action at  $h$  is uncertain whether he is at  $x$  or at some other  $x' \in h(x)$ . We require that if  $x' \in h(x)$ , then the same player moves at  $x$  and  $x'$ . Otherwise, players may disagree who was supposed to move.

Information sets partition the decision-nodes such that each node belongs to exactly one information set and no more. It is in this way that information sets are related to histories. As you can see in the example, it is perfectly fine to have information sets with more than one decision node. However, it is impossible for the same decision node to appear in more than one information set.

Recall that the action sets are defined in terms of information sets. That is,  $A_i(h)$  is the set of actions from which player  $i$  may choose at information set  $h$ . It is essential to realize that this implies that for all nodes in this information set, the actions available at each are the same. That is, if  $x' \in h(x)$ , then  $A_i(x') = A_i(x)$ . Thus, we can let  $A_i(h)$  denote the action set at information set  $h$ .

To see why this must be the case, suppose that player 2 had another option, say "punt", at the node reached by the history  $h_3 = (\text{black}, R)$  that was not available after history  $h_4 = (\text{red}, r)$ . This means that she could punt if and only if player 1 had a black card. But how would player 2 exercise this option if she does not know the color of the card? To represent this situation, we would have to give player 2 an action called "try to punt" and add it to both nodes in her information set. Then, if she chooses this option, she would succeed when the card is black but fail when it is red.

Note, on the other hand, that we could easily give player 1 different actions (or numbers of actions) at each of his nodes 1.b and 1.c because they belong to different information sets. It is to emphasize this that I label the actions differently in Fig. 3 (p. 5), with lowercase and uppercase letters, depending on the color.

The point is that if a player has two nodes with different sets of actions, then these nodes cannot belong to the same information set. However, one can easily have different nodes with the same sets of actions even though the nodes are not in the same information set.

This completes the extensive form representation of the card game. Note that we have specified the players, the rules of the game (who moves when and what options they have), the outcomes in terms of

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the information obtained from observing raising and knowledge of player 1's optimal strategy. In some games, the uncertainty will be fully resolved—even though player 2 cannot observe what is known to the opponent, she can *infer* that information from his observable behavior and knowledge that he, being intelligent and rational, is choosing his optimal strategy. Of course, player 1 knows all of that full well, so he may well try to obfuscate her inferences, just as he will do in this particular game. His optimal strategy is to prevent this inference. Even then player 2 will be able to learn something from the fact that he's chosen to raise. Observe, incidentally, that unless you assume that players pursue the best strategies to the best of their abilities, you cannot make such inferences, and behavior becomes unintelligible. Among other things, this would imply that we simply cannot perform any sort of meaningful analysis as social scientists.

terminal histories, the payoffs associated with these outcomes, the information available to the players when they move, and the probability distribution of the chance events.

### 1.1 Formal Definition of the Extensive Form

In most applications, the game trees would rarely be drawn, and so one must make do with the mathematical description of the extensive form. It is necessary to go through this exercise to understand the methodology of this fundamental class of games. We shall rarely, if ever, need to resort to the finer detail, but the mathematical description allows us to define two important categories of games (perfect and imperfect recall), of which we shall only study one. The following definition follows Fudenberg & Tirole (1991).

DEFINITION 1. The extensive form of a game,  $\Gamma = \{\mathcal{I}, (X, \succ), \iota(\cdot), A(\cdot), H, u\}$ , contains the following elements:

1. A set of players denoted by  $i \in \mathcal{I}$ , with  $\mathcal{I} = \{N, 1, 2, \dots\}$ , with  $N$  representing the pseudo-player Nature;
2. A tree,  $(X, \succ)$ , which is a finite collection of nodes  $x \in \mathcal{X}$  endowed with the precedence relation  $\succ$ , where  $x \succ x'$  means “ $x$  is before  $x'$ .” This relation is transitive and asymmetric, and thus constitutes a *partial order*.<sup>4</sup> This rules out cycles where the game may go from node  $x$  to a node  $x'$ , from  $x'$  back to  $x$ .<sup>5</sup> In addition, we require that each node  $x$  has exactly one immediate predecessor, that is, one node  $x' \succ x$  such that  $x'' \succ x$  and  $x'' \neq x'$  implies  $x' \succ x''$  or  $x'' \succ x'$ . Thus, if  $x'$  and  $x''$  are both predecessors of  $x$ , then either  $x'$  is before  $x''$  or  $x''$  is before  $x'$ .
3. A set of terminal nodes, denoted by  $z \in \mathcal{Z}$  consisting of all nodes that are not predecessors of any other node. Because each  $z$  determines the path through the tree, it represents an outcome of the game. The payoffs for outcomes are assigned by the Bernoulli payoff functions  $u_i : \mathcal{Z} \rightarrow \mathbb{R}$ , and  $u = (u_1(\cdot), \dots, u_I(\cdot))$  is the collection of these functions, one for each player.
4. A map  $\iota : \mathcal{X} \rightarrow \mathcal{I}$ , with the interpretation that player  $\iota(x)$  moves at node  $x$ . A function  $A(x)$  that denotes the set of feasible actions at  $x$ .
5. Information sets  $h \in H$  that partition the nodes of the tree such that every node is exactly in one set. The interpretation of  $h(x)$  is that information set  $h$  contains the node  $x$ . We require that if  $x' \in h(x)$ , then  $A(x') = A(x)$ , and so we can let  $A(h)$  denote the set of feasible actions at information set  $h$ .
6. A probability distribution over the set of alternatives for all chance nodes.

This definition now allows us to make several ideas very precise.

### 1.2 Perfect Recall

We shall require that players have *perfect recall*. That is, a player never forgets information he once knew, and each player knows the actions he has chosen previously. (As we shall see, the fact that players may know all previous history does not force us to assume that he will take it all into account when making decisions.) This is accomplished by requiring that:

<sup>4</sup>It is not a complete order because two nodes may not be comparable. For example, consider player 2's information set in Fig. 3 (p. 5): Neither of the nodes precedes the other.

<sup>5</sup>To see this, suppose we constructed a game such that  $x \succ x' \succ x$ . By transitivity,  $x \succ x$ , but this violates asymmetry.

- A) if two decision nodes are in the same information set, then neither is a predecessor of the other; and
- B) if two nodes  $x'$  and  $x''$  are in the same information set and one of them has a predecessor  $x$ , then the other one has a predecessor  $\hat{x}$  (possibly  $x$  itself) in the same information set as  $x$  and the action taken at  $x$  that leads to  $x'$  is the same as the action taken from  $\hat{x}$  that leads to the  $x''$ .

The games in Fig. 4 (p. 7) illustrate some cases of imperfect recall that this requirement eliminates.

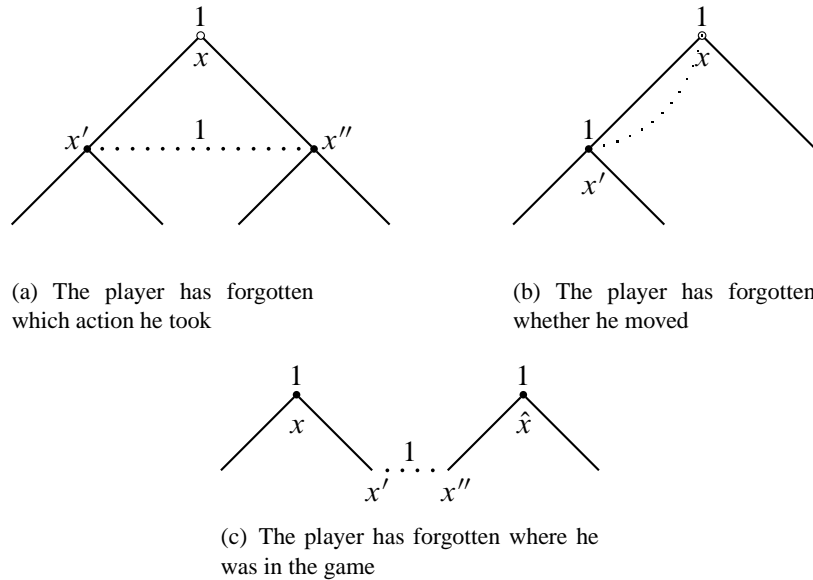


Figure 4: Games of Imperfect Recall.

The situation in Fig. 4(a) (p. 7) is ruled out by condition Condition B because even though both  $x'$  and  $x''$  in player 1's second information set have the same predecessor,  $x$ , the actions leading from  $x$  to the information set are different. The situation in Fig. 4(b) (p. 7) is ruled out by Condition A because  $x$  and  $x'$  are in the same information set but  $x$  is a predecessor of  $x'$ . Finally, the situation in Fig. 4(c) (p. 7) is ruled out by Condition B: because  $x'$  and  $x''$  are in the same information set and even though  $x$  is a predecessor of  $x'$  and  $\hat{x}$  is a predecessor of  $x''$ ,  $x$  and  $\hat{x}$  are not in the same information set themselves.

The literature on games with imperfect recall is very small, although there are some very interesting papers that might be worth looking at (e.g. the famous game where a drunk driver forgets whether he's been past an exit on the freeway). These games are still quite exotic and their application has been of limited usefulness. This is not to say that there are no exciting areas where these can be applied. One interesting area of research is machine game models of repeated situations: these machines have limited memory and since information is costly to acquire, a player may "forget" some of his past actions. This approach has been extensively used in low-rationality models of learning (evolutionary game theory, for example), where players look at a most recent past when forming expectations about future behavior. This course will only deal with games of perfect recall.

### 1.3 Finite and Infinite Games

There are three different conceptions of finiteness buried in the definition of extensive form games. The mathematical description can be easily extended to cover these as well.

DEFINITION 2. A **finite** game has (i) a finite number of players, (ii) a finite number of actions, and (iii) finite length histories. Otherwise, the game is **infinite**.

Note that relaxing any of the three requirements results in an infinite number of nodes. Thus, a game is finite if it has a finite number of nodes. Some examples of useful infinite games that we shall encounter include games where players choose actions from some interval that is a subset of the real line; games which can be repeated indefinitely; or games involving an infinite number of players (we shall see how these games are a way to model incomplete information).

## 1.4 Informational Categories

We now make very precise several different informational categories. Make sure you understand the terms because we shall use them quite a bit.

DEFINITION 3. We distinguish the following informational categories:

- A game is one of **perfect information** if each information set is a singleton; otherwise it is a game of **imperfect information**.
- A game is one of **certainty** if it has no moves by Nature; otherwise it is a game of **uncertainty**.
- A game is one of **complete information** if all payoff functions are common knowledge; otherwise it is a game of **incomplete information**.
- A game is one of **symmetric information** if no player has information that is different from other players when he moves or at the terminal nodes; otherwise it is a game of **asymmetric information**.

Myerson's Card Game shown in Fig. 3 (p. 5) is a game of complete but imperfect (and asymmetric) information that is also one of uncertainty. Games of imperfect recall are always games of imperfect information.

We shall see games of incomplete (asymmetric) information later in the course. We shall also see how they can be modeled (and solved) as games of imperfect information. It is worth noting that although many games of incomplete information are also games of asymmetric information, the two concepts are not equivalent. For example, the famous principal-agent problem has complete but asymmetric information: both players know all payoff functions but the principal does not observe the agent's effort, even after the end of the game.

It is also possible to have games of incomplete but symmetric information. For example, a Prisoners' Dilemma where Nature moves first and randomly assigns different payoffs to the outcomes, unknown to either player.

## 2 Strategies in EFG

### 2.1 Pure Strategies

Player  $i$ 's **strategy**,  $s_i$ , is a complete rule of action that tells him which action  $a_i \in A_i$  to choose at each of his information sets. That is, a strategy specifies what the player is going to do every time it is his turn to move given what he knows. A player's **strategy space** (sometimes also called a *strategy set*),  $S_i = \{s_i\}$ , is the set of all possible strategies.

A strategy is a **complete contingent plan of action**. That is, a strategy in an extensive form game is a plan that specifies the action chosen by the player for *every* history after which it is his turn to move, that is, at *each* of his information sets. This is a bit counter-intuitive because it means that the strategy must specify moves at information sets that might never be reached because of actions specified by the player's strategy at earlier information sets.



DEFINITION 4. Let  $\Gamma$  be a game in extensive form. A **pure strategy** for player  $i \in \mathcal{I}$  is a function  $s_i : \mathcal{H} \rightarrow \mathcal{A}$  such that  $s_i(h) \in A_i(h)$  for all  $h \in \mathcal{H}$ .

Let's list the strategies for the two players in Myerson's Card Game in Fig. 3 (p. 5). Player 1 has two information sets, labeled "b" and "c", with  $A_1(b) = \{R, F\}$  and  $A_1(c) = \{r, f\}$ , so his strategy must specify two actions,  $a_b \in A_1(b)$  and  $a_c \in A_1(c)$ . We shall write his strategy as an ordered set:  $s_1 = (a_b, a_c)$ , with the first element denoting the action to take at information set "b" and the second denoting the action to take at information set "c". This gives four pure strategies for player 1:

$$S_1 = \{(R, r), (R, f), (F, r), (F, f)\}.$$

For example,  $(R, f)$  is the strategy "raise if the card is black, and fold if the card is red."

Player 2 knows that she won't see the color and will only get to choose if player 1 raises, in which case she will either have to meet or pass. There is only one information set for player 2, so her pure strategy must simply specify the action,  $a_0 \in A_2(0) = \{m, p\}$ , she is to take at this information set. Thus,

$$S_2 = \{m, p\}.$$

The strategy  $m$  is then "meet if player 1 raises."

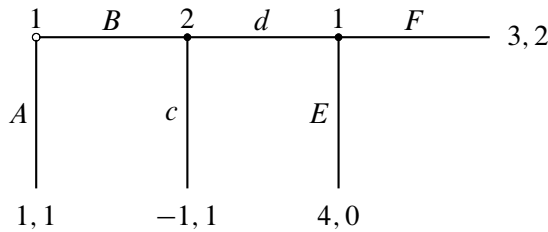


Figure 5: EFG With Two Info Sets for Player 1.

Consider now the game in Fig. 5 (p. 9). It has two players,  $i \in \{1, 2\}$ . The game also has seven histories:  $H = \{(\emptyset), (A), (B), (B, c), (B, d), (B, d, E), (B, d, F)\}$ . Recall that  $\mathcal{H}_i$  denotes the set of information sets for player  $i$ , and  $A_i(h)$  denotes the set of available actions at information set  $h$  for all  $h \in \mathcal{H}_i$ . At the information set  $\emptyset$ , player 1 has two actions available:  $A_1(\emptyset) = \{A, B\}$ . At the information set  $(B, d)$ , he has two actions available  $A_1(B, d) = \{F, E\}$ . Player 2 only gets to move at the information set  $B$ , and has two actions available there:  $A_2(B) = \{c, d\}$ . There are four terminal histories:  $Z = \{(A), (B, c), (B, d, E), (B, d, F)\}$ .

Since a strategy is a complete contingent plan of action, it must specify the actions to be taken at every information set. Player 1 has two information sets in the game, and therefore his strategy will have 2 components: an action to take at the first information set, and an action to take at the second information set. Since in both cases he has two actions available, he has a total of four different strategies:

$$S_1 = \{(AE), (AF), (BE), (BF)\}.$$

Player 2 has only one information set, with two actions there, and so she has only two possible strategies:

$$S_2 = \{c, d\}.$$

This game illustrates a point that is worth emphasizing. It is extremely important to remember that a strategy specifies the action chosen by a player for *every* information set at which it is his turn to move, *even for information sets that are never reached if the strategy is followed*. That is, in the game in Fig. 5

(p. 9), the first two strategies,  $(AE)$  and  $(AF)$  specify actions after the history  $(B, d)$  even though they specify action  $A$  at the initial node (which means that when the strategy is followed, history  $(B, d)$  will never be realized, and the second information set will never be reached). In this sense, a strategy differs from what we naturally consider a plan of action. In this instance, every-day language is misleading. We may say that we “plan to choose  $B$ ” and since the game will end, there is no reason to plan what to do if we played  $A$  instead. However, here we want to know whether  $B$  is better than  $A$  for player 1. To decide whether this is the case, we need to know what the consequences of choosing  $A$  are (otherwise we cannot compare the two actions). But to evaluate the consequences of  $A$ , we need to take into account what he would optimally do at his last information set and incorporate this into player 2’s expectations to infer what she will do at her information set. Choosing  $B$  can only be optimal *in the context of expectations about what would happen if the player chose action  $A$  instead*. It is because we want to find optimal strategies that we must engage in these comparisons and it is for that reason that we must specify the full strategy in what appears to be a redundant fashion. This will become clearer when we analyze some games later on.

A **strategy profile**,  $s = (s_1, s_2, \dots, s_n)$ , is an ordered set of strategies consisting of one strategy for each of the  $n$  players in the game. One extraordinarily useful piece of notation can let us focus on player  $i$ ’s strategy  $s_i$  in the profile  $s$ . We can partition the strategy profile  $s$  as:

$$(s_i, s_{-i}) \equiv s,$$

where  $s_i$  is player  $i$ ’s strategy, and  $s_{-i}$  is the set of strategies for all other players. For example, if  $s = (s_1, s_2, s_3, s_4, s_5)$ , and we specify  $(s_i, s_{-i})$  for player  $i = 3$ , then  $s_i = s_3$ , and  $s_{-i} = (s_1, s_2, s_4, s_5)$ . Let  $S = S_1 \times S_2 \times \dots \times S_n$  denote the set of strategy profiles.

Because a strategy profile specifies what each player is going to do at every point in the game where it is his turn to move, it in effect describes *how* the game will be played and what its outcome will be if the players follow the strategies in the profile. In other words, each strategy profile will yield:

- one outcome if that there are no moves by chance; or
- a probability distribution over outcomes if there are moves by chance and the strategies are consistent with information sets where Nature moves.

Some people define players’ preference orderings over strategy profiles, but I find this confusing even though it is equivalent to defining them over outcomes. It is confusing because one may think that players actually care about the strategies being played apart from the outcomes they produce. (If this is the case, then this fact must be reflected in the payoffs associated with the outcomes.) We shall define them over outcomes. A player’s payoff,  $u_i(s)$ , is the expected utility that player  $i$  receives from the outcome produced by the strategy profile  $s \in S$ . Thus, each player  $i$ ’s goal in a game is to choose  $s_i \in S_i$  that maximizes  $u_i(s_i, s_{-i})$ .

## 2.2 Mixed Strategies in EFG

The definition of mixed strategies in EFG is exactly the same as the definition in strategic form games. To summarize,

**DEFINITION 5.** A **mixed strategy** for player  $i$ , denoted by  $\sigma_i$ , is a probability distribution over  $i$ ’s set of pure strategies  $S_i$ . Denote the mixed strategy space for player  $i$  by  $\Sigma_i$ , where  $\sigma_i(s_i)$  is the probability that  $\sigma_i$  assigns to the pure strategy  $s_i \in S_i$ . The space of mixed strategy profiles is denoted by  $\Sigma = \Delta \Sigma_i$ .

### 2.3 Behavior Strategies in EFG

Unlike strategic form games, extensive form games admit two distinct types of randomization: a player can either randomize over his pure strategies or he can randomize over the actions at each of his information sets. The second type of randomizing strategy is the **behavior strategy**, which specifies a probability distribution over actions at each information set. *These distributions are independent.* That is, a behavior strategy specifies the probabilities with which actions are chosen at every information set. Thus, a pure strategy is a special kind of behavior strategy where the distribution at each information set is degenerate.

To help illustrate the difference between the two types of randomization, Luce and Raiffa (1957) offer the following analogy: A pure strategy is a book of instructions, where each page tells how to play at a particular information set. The space of pure strategies is a library of these books. A mixed strategy is a probability distribution over this library (i.e. it specifies the probability with which books are chosen). A behavior strategy is a single book where each page prescribes a random action. Thus, a player may randomly select a pure strategy or he might plan a set of randomizations, one for every point at which he has to take action.

An example may be helpful. Consider the game in Fig. 5 (p. 9) and recall that player 1 has four pure strategies:  $(AE)$ ,  $(AF)$ ,  $(BE)$ , and  $(BF)$ . A mixed strategy is a probability distribution over these four strategies. For example, a mixed strategy  $\sigma = (1/4, 1/4, 1/4, 1/4)$  specifies that player 1 will play each of his pure strategies with equal probability of  $1/4$ . Another mixed strategy might be  $\sigma = (1/3, 0, 1/6, 1/2)$ , which specifies that player 1 should play  $AE$  with probability  $1/3$ ,  $AF$  with probability 0,  $BE$  with probability  $1/6$ , and  $BF$  with probability  $1/2$ . You can see the close correspondence with mixed strategies in normal form games.

On the other hand, a behavior strategy for player 1 would specify probabilities for actions at all information sets. Because player 1 has two information sets, the strategy must specify two probability distributions, one for each information set. For example,  $\beta = (1/4, 1/4)$  means that player 1 will choose  $A$  at his first information set with probability  $1/4$  (and choose  $B$  with complementary probability  $3/4$ ), and he will choose  $E$  with probability  $1/4$  at the second information set. Another behavior strategy might be  $\beta = (0, 1/2)$ , which specifies that player 1 should choose  $B$  with probability 1 at the first information set and play  $E$  and  $F$  with equal probability at the second information set. Just like a pure strategy will have as many elements as there are information sets at which the player must move, the behavior strategy will also have as many elements as there are information sets. The difference is that the pure strategy will prescribe a certain action for each information set whereas the behavior strategy prescribes a probability distribution over the actions at this set. (Of course, the number of elements in a mixed strategy equals the number of pure strategies.) As we noted, a pure strategy is a behavior strategy with degenerate distributions at each information set. So, for example, the pure strategy  $BE$  is the behavior strategy  $\beta = (0, 1)$  just as it is the degenerate mixed strategy  $\sigma = (0, 0, 1, 0)$ .

### 2.4 Equivalence of Mixed and Behavior Strategies

As you probably already suspect, the two types of randomizing strategies are closely related. We shall call two strategies *equivalent* if they induce the same probability distributions over outcomes for all strategies of the opponents.<sup>6</sup> Intuitively, two strategies are equivalent if they have the same consequences regardless of what the other players do.

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<sup>6</sup>This is the same concept of equivalence we used when we discussed the reduced normal form representation of extensive games in the previous section.

### 2.4.1 Mixed Strategy Equivalent to a Behavior Strategy

Let's see how we can generate a mixed strategy that is equivalent to some arbitrary behavior strategy  $\beta_i$  for player  $i$ . Let  $\beta_i(h_i)(a_i)$  denote the probability with which action  $a_i \in A_i(h_i)$  is taken (that is the probability with which an action is chosen from the set of actions available after history  $h_i$ ). Let  $s_i(h_i)$  denote the action specified by the pure strategy  $s_i$  at the information set  $h_i$  (and so  $s_i$  specifies one action for all information sets where player  $i$  gets to move). Define the mixed strategy  $\sigma_i$  to assign the following probability to each pure strategy  $s_i$ :

$$\sigma_i(s_i) = \prod_{h_i \in H} \beta_i(h_i)(s_i(h_i)). \quad (1)$$

That is, the probability with which the pure strategy is chosen is simply the product of probabilities assigned by the behavior strategy to the action the pure strategy prescribes at each information set. Note that we made use of the assumption that the behavior randomizations are independent across information sets.<sup>7</sup>

Let's ask ourselves about the intuition behind this. Essentially, a pure strategy,  $s_i$ , gives a "path" of play through the game: given what other players are doing, this strategy tells  $i$  what to choose at each of his information sets until the game tree reaches a terminal node. This means that  $\sigma_i$  would have to assign to that "path" a probability that equals the probabilities with which each of its separate components is taken by  $i$ 's choice. Since  $\beta_i$  gives the probability of the action prescribed by  $s_i$  for each information set, the probability of the entire "path" is just the product of the probabilities that  $i$  picks the relevant actions that constitute that path.

Consider the (Little Horsey) game in Fig. 5 (p. 9). A behavior strategy for player 1 has two elements, a probability distribution over his two actions  $\{A, B\}$  at his first information set, and another probability distribution over the actions  $\{E, F\}$  at his second information set. Consider some fixed (possibly mixed) strategy for player 2,  $\sigma_2$  such that  $\sigma_2(d) > 0$ , and consider the outcome after history  $(B, d, F)$ . Denote this outcome by  $z_4$ . The only pure strategy for player 1 that can produce this with positive probability is  $s_1 = (B, F)$ . That is  $\Pr[z_4|s_1] = \sigma_2(d)$ . Observe now that a (non-degenerate) behavior strategy will put positive probabilities on both  $B$  and  $F$  but will not choose them with certainty. Hence, the probability of  $z_4$  will be  $\Pr[z_4|\beta_1] = \beta_1(\emptyset)(B) \times \sigma_2(d) \times \beta_1(Bd)(F)$ . That is, it multiplies the probabilities it assigns to the actions specified by  $s_1$  at each information set:  $\Pr[z_4|\beta_1] = \beta_1(\emptyset)(s_1(\emptyset)) \times \sigma_2(d) \times \beta_1(Bd)(s_1(Bd))$ , where we note that  $s_1 = (B, F)$  is, if we were to use the full definition of a pure strategy as a function that takes an information set and returns an action, equivalent to  $s_1(\emptyset) = B$  and  $s_1(Bd) = F$ . Now, a mixed strategy for player 1 can also produce  $z_4$  with positive probability as long as  $\sigma_1(BF) > 0$ . In particular, since we want  $\sigma_1$  to produce  $z_4$  with the same probability as  $\beta_1$ , it must be the case that in that mixed strategy the probability of player 1 choosing *both*  $B$  and  $F$  at the respective information sets must be the same under  $\sigma_1$  as it is under  $\beta_1$ . Under  $\beta_1$ , we have seen that the probability of choosing  $B$  and  $F$  is  $\beta_1(\emptyset)(B) \times \beta_1(Bd)(F)$ , which would give  $z_4$  with probability  $\sigma_2(d)$ . Since the only way to reach this outcome must involve playing  $s_1$ , the mixed strategy must assign this exact probability to that pure strategy:  $\sigma_1(s_1) = \beta_1(\emptyset)(s_1(\emptyset)) \times \beta_1(Bd)(s_1(B, d))$ , that is, exactly as in (1). The probability of reaching  $z_4$  using  $\sigma_1$  is also  $\sigma_2(d)$ .

Let's now consider a specific example. To check equivalence, we first need to specify the distribution over outcomes. The Little Horsey game in Fig. 5 (p. 9) has four outcomes. Let the probability distribution  $(z_1, z_2, z_3, z_4)$  denote the associated probabilities for the outcomes  $(1, 1)$ ,  $(-1, 1)$ ,  $(3, 2)$ , and  $(4, 0)$ . Finally, let  $\sigma_2(c)$  denote the probability with which player 2 chooses  $c$  and  $\sigma_2(d) = 1 - \sigma_2(c)$  denote the probability with which she chooses  $d$ . The behavior strategy  $\beta = ((1/4, 3/4), (1/4, 3/4))$ ,

<sup>7</sup>This holds for all games of perfect recall. In games of imperfect recall, it is possible to have behavior strategies that cannot be duplicated by any mixed strategy.

where player 1 chooses  $A$  and  $E$  with probability  $1/4$ , induces the probability distribution over outcomes  $(1/4, 3/4\sigma_2(c), 3/16\sigma_2(d), 9/16\sigma_2(d))$ . (We obtained the probabilities for  $z_3$  and  $z_4$  by multiplying the probability of each action specified by the behavior strategy by the probability that the initial action is  $B$ . You should verify that the distribution over outcomes is valid: i.e. all probabilities sum to 1.) Now, using our Equation 1, we can define the mixed strategy  $\sigma$  as follows:

$$\begin{aligned}\sigma(AE) &= \beta(\emptyset)(A) \times \beta(Bd)(E) = 1/4 \times 1/4 = 1/16 \\ \sigma(AF) &= \beta(\emptyset)(A) \times \beta(Bd)(F) = 1/4 \times 3/4 = 3/16 \\ \sigma(BE) &= \beta(\emptyset)(B) \times \beta(Bd)(E) = 3/4 \times 1/4 = 3/16 \\ \sigma(BF) &= \beta(\emptyset)(B) \times \beta(Bd)(F) = 3/4 \times 3/4 = 9/16\end{aligned}$$

(We again verify that this is a valid probability distribution by noting that the probabilities all sum to 1.) Is this mixed strategy equivalent to the original behavior strategy? That is, does it induce the same probability over outcomes regardless of what the other player does? The probability of outcome  $z_1$  equals the probability that player 1 chooses  $A$ , which he does in two of his strategies, and so it is  $\sigma(AE) + \sigma(AF) = 1/4$ . The probability of  $z_2$  is the probability that player 1 will choose  $B$ , which is  $\sigma(BE) + \sigma(BF) = 3/4$ , multiplied by the probability that player 2 chooses  $c$ . This yields  $3/4\sigma_2(c)$ . The probability of  $z_3$  is the probability that player 1 chooses both  $B$  and  $E$  multiplied by the probability that player 2 chooses  $d$ , which yields  $\sigma(BE)\sigma_2(d) = 3/16\sigma_2(d)$ . Finally, the probability of  $z_4$  is the probability that player 1 chooses both  $B$  and  $F$ ,  $\sigma(BF)$ , multiplied by the probability that player 2 chooses  $d$ , which yields  $9/16\sigma_2(d)$ . To summarize, the probability distribution over outcomes induced by the mixed strategy  $\sigma$  as defined above is  $(1/4, 3/4\sigma_2(c), 3/16\sigma_2(d), 9/16\sigma_2(d))$ , which is the same as the probability distribution induced by the behavior strategy  $\beta$ . We have now seen how to generate an equivalent mixed strategy from an arbitrary behavior strategy. But there is more to equivalence than this!

## 2.4.2 Equivalence Theorem

An important result is that in a game of perfect recall, mixed and behavior strategies are equivalent.

THEOREM 1 (KUHN 1953). *In a game of perfect recall,*

- every behavior strategy is equivalent to every mixed strategy that generates it;
- every mixed strategy is equivalent to the unique behavior strategy it generates. □

That is, different mixed strategies can generate the same behavior strategy even though each mixed strategy either generates exactly one behavior strategy or else infinitely many behavior strategies. To make this a bit more concrete, two different mixed strategies can generate the same behavior strategy (we shall see an example below). The first part of the claim is that this behavior strategy is going to be equivalent to each of the two different mixed strategies that generate it. The two mixed strategies are *behaviorally equivalent*.

Further, every mixed strategy has at least one behavioral representation, and it may have many. It may have many if there are information sets that the mixed strategy does not reach with positive probability: In this case it does not matter what probability distribution the behavior strategy specifies for that information set. If, however, the mixed strategy reaches all information sets with positive probability, then it will generate a unique behavior strategy. The second part of the claim states the these will be equivalent.

Finally, note that we can generate a mixed strategy  $\sigma_i$  from a behavior strategy  $\beta_i$  as shown above in (1). In this case,  $\sigma_i$  is the mixed representation of  $\beta_i$ , and they are equivalent. Further, it is not hard to show that if  $\sigma_i$  is the mixed representation of  $\beta_i$ , then  $\beta_i$  is the behavioral representation of  $\sigma_i$ .

To see how the theorem works, let's derive a behavior strategy for some given mixed strategy. Let  $\sigma_i$  be a mixed strategy for player  $i$ . For any history  $h_i$ , let  $R_i(h_i)$  denote the set of player  $i$ 's pure strategies that are consistent with  $h_i$ . That is, for all  $s_i \in R_i(h_i)$ , there is a profile  $s_{-i}$  for the other players that reaches  $h_i$ . We shall call the strategies in  $R_i(h_i)$  *consistent* with the history  $h_i$ . For example, in the Little Horsey game from Fig. 5 (p. 9), all four pure strategies for player 1 are consistent with his first information set,  $\emptyset$  for the simple reason that the initial information set is always reached regardless of what players are going to do from that point on. On the other hand, the information set  $(Bd)$  can only be reached for some strategy by player 2 (in this case,  $d$ ) provided player 1 chooses  $B$  at his first information set. There are only two pure strategies that involve such a choice:  $(BE)$  and  $(BF)$ . Therefore,  $R_1(Bd) = \{BE, BF\}$ , and neither  $AE$  nor  $AF$  is consistent with the history  $Bd$ .

Now let  $\pi_i(h_i)$  be the sum of probabilities according to  $\sigma_i$  of all the pure strategies that are consistent with  $h_i$ :

$$\pi_i(h_i) = \sum_{s_i \in R_i(h_i)} \sigma_i(s_i).$$

Intuitively, this is the probability with which the game will reach  $h_i$  provided  $i$  (and the other players) choose actions consistent with this history. It answers the question: "Suppose all other players use pure strategies that are on the path toward  $h_i$ . What is the probability of reaching  $h_i$  if player  $i$  uses  $\sigma_i$ ?" In our example,  $\pi_1(\emptyset) = \sigma_1(AE) + \sigma_1(AF) + \sigma_1(BE) + \sigma_1(BF) = 1$ , and  $\pi_1(Bd) = \sigma_1(BE) + \sigma_1(BF)$ . In either case, we are supposing that player 2 is choosing  $d$  in the sense that she is not playing a strategy that would make reaching  $Bd$  impossible no matter what player 1 does.

Let  $\pi(h_i, a_i)$  denote the sum of probabilities according to  $\sigma_i$  of all pure strategies that are consistent with  $h_i$  followed by action  $a_i \in A_i(h_i)$ . So we have

$$\pi_i(h_i, a_i) = \sum_{s_i \in R_i(h_i) \wedge s_i(h_i) = a_i} \sigma_i(s_i).$$

Intuitively, this is very similar to  $\pi_i(h_i)$  except that it asks "What is the probability of reaching  $h_i$  and choosing  $a_i$  at that information set?" (Again, provided the other players use strategies that do not preclude reaching that point in the game.) In our example,  $\pi_1(\emptyset, A) = \sigma_1(AE) + \sigma_1(AF)$  because each of  $AE$  and  $AF$  is both consistent with the initial history  $\emptyset$  and prescribes  $A$  as the action at that set. Similarly,  $\pi_1(\emptyset, B) = \sigma_1(BE) + \sigma_1(BF)$ . At the second information set, we have  $\pi_1(Bd, E) = \sigma_1(BE)$  because even though both  $BE$  and  $BF$  are consistent with this history, only  $BE$  involves choosing  $E$  at the second information set. Analogously,  $\pi_1(Bd, F) = \sigma_1(BF)$ .

We now have the two components we need. Observe that  $\pi_i(h_i, a_i)$  is the probability of reaching  $h_i$  and playing  $a_i$ . However, to define  $\beta(h_i)(a_i)$ , we need to find the probability of playing  $a_i$  *provided*  $h_i$  has been reached. This requires us to condition  $\pi_i(h_i, a_i)$  on the probability of reaching  $h_i$ , which is  $\pi_i(h_i)$ . If  $\sigma_i$  assigns positive probability to some  $s_i \in R_i(h_i)$ , define the probability that the behavior strategy  $\beta_i$  assigns to  $a_i \in A_i(h_i)$  as the probability of taking action  $a_i$  conditional on reaching the information set  $h_i$ :

$$\beta_i(h_i)(a_i) = \frac{\pi_i(h_i, a_i)}{\pi_i(h_i)}.$$

Intuitively, the probability of picking  $a_i$  at the information set  $h_i$  is the probability of reaching  $h_i$  and picking  $a_i$  conditioned on the probability of reaching  $h_i$ . In our example,  $\beta_1(\emptyset)(A) = \sigma_1(AE) + \sigma_1(AF)$  and  $\beta_1(\emptyset, B) = \sigma_1(BE) + \sigma_1(BF)$ . At the second information set,

$$\beta_1(Bd, E) = \frac{\sigma_1(BE)}{\sigma_1(BE) + \sigma_1(BF)};$$

that is, the probability the behavior strategy must assign to the action  $E$  is the probability  $\sigma_1$  assigns to it conditional on reaching this information set if  $\sigma_1$  is followed. Finally,

$$\beta_1(Bd, F) = \frac{\sigma_1(BF)}{\sigma_1(BE) + \sigma_1(BF)}.$$

How we define  $\beta_i(h_i)(a_i)$  if  $\pi_i(h_i) = 0$  is immaterial.<sup>8</sup> One possible specification is to assign the probabilities given by the mixed strategy:  $\beta_i(h_i)(a_i) = \sum_{s_i(h_i)=a_i} \sigma_i(s_i)$ , but anything will do. In either case, the  $\beta_i(\cdot)(\cdot)$  are nonnegative, and

$$\sum_{a_i \in A_i(h_i)} \beta_i(h_i)(a_i) = 1,$$

because each  $s_i$  specifies an action for player  $i$  at the information set  $h_i$ . In other words,  $\beta_i$  specifies a valid distribution for each information set  $h_i$ . If  $\pi_i(h_i) > 0$  for all histories, then the mixed strategy will generate a unique behavior strategy.

Let's look at concrete example. Consider the game in Fig. 6 (p. 15). We want to find the behavior strategy for player 1 that is equivalent to his mixed strategy in which he plays  $(B, R)$  with probability 0.4,  $(B, L)$  with probability 0.1, and  $(A, L)$  with probability 0.5.

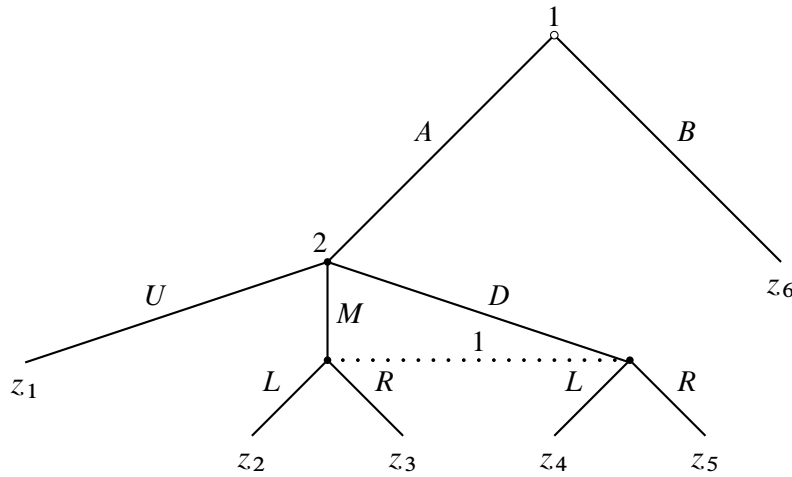


Figure 6: A Game for Kuhn's Theorem, I.

We have  $\sigma_1(B, R) = 0.4$ ,  $\sigma_1(B, L) = 0.1$ ,  $\sigma_1(A, L) = 0.5$ , and (since the mixed strategy is a probability distribution),  $\sigma_1(A, R) = 0$ . Player 1 has two information sets: one after the  $\emptyset$  history, and another after the histories  $(A, M)$  and  $(A, D)$ . The behavior strategy will thus specify two probability distributions, one for each information set.

Since  $h_1 = \emptyset$  is the initial history, all pure strategies are consistent with it. (This is trivially true: there is no pure strategy for player  $i$  such that this history cannot be reached.) Thus,

$$R_1(h_1) = \{(A, L), (A, R), (B, L), (B, R)\},$$

which also means  $\pi_1(h_1) = 1$ . Since there are two possible actions player 1 can take at  $h_1$ , we must calculate  $\pi_1(h_1, A)$  and  $\pi_1(h_1, B)$ . There are two pure strategies  $s_1$  such that  $s_1 \in R_1(h_1) \wedge s_1(h_1) = A$ , and these are  $(A, L)$  and  $(A, R)$ . Therefore,  $\pi_1(h_1, A) = \sigma_1(A, L) + \sigma_1(A, R) = 0.5$ . Also, there are

<sup>8</sup>Since  $h_i$  cannot be reached under  $\sigma_i$ , the behavior strategies at  $h_i$  are arbitrary in the same sense that Bayes' Rule does not determine posterior probabilities after 0-probability events.

two pure strategies such that  $s_1 \in R_1(h_1) \wedge s_1(h_1) = B$ , and these are  $(B, L)$  and  $(B, R)$ . This means  $\pi_1(h_1, B) = \sigma_1(B, L) + \sigma_1(B, R) = 0.5$ . We now have  $\beta_1(h_1)(A) = \pi_1(h_1, A)/\pi_1(h_1) = 0.5/1 = 0.5$  and also  $\beta_1(h_1)(B) = \pi_1(h_1, B)/\pi_1(h_1) = 0.5/1 = 0.5$ .<sup>9</sup> So,  $\beta_1(h_1)(A) = \beta_1(h_1)(B) = 0.5$ .

Now consider  $h_2 = \{(A, M), (A, D)\}$ . The only pure strategies for player 1 that are consistent with this history are the ones that specify  $A$  for the move at the first information set. (That is, there exists no strategy for player 2 such that  $h_2$  is reached if player 1 chooses  $B$  at the first information set.) Therefore,  $R_1(h_2) = \{(A, L), (A, R)\}$ , which means that  $\pi_1(h_2) = \sigma_1(A, L) + \sigma_1(A, R) = 0.5$ . Since player 1 has two possible actions at  $h_2$ , we must also calculate  $\pi_1(h_2, L)$  and  $\pi_1(h_2, R)$ . There is only one pure strategy such that  $s_1 \in R_1(h_2) \wedge s_1(h_2) = L$ , and it is  $(A, L)$ . Therefore,  $\pi_1(h_2, L) = \sigma_1(A, L) = 0.5$ . Also, there is only one pure strategy such that  $s_1 \in R_1(h_2) \wedge s_1(h_2) = R$ , and it is  $(A, R)$ , which means  $\pi_1(h_2, R) = \sigma_1(A, R) = 0$ . We now have  $\beta_1(h_2)(L) = \pi_1(h_2, L)/\pi_1(h_2) = 0.5/0.5 = 1$ , and we also have  $\beta_1(h_2)(R) = \pi_1(h_2, R)/\pi_1(h_2) = 0/0.5 = 0$ .<sup>10</sup>

We conclude that the mixed strategy  $\sigma_1$  has an equivalent behavior strategy  $\beta_1$ , which is as follows:

$$\begin{aligned}\beta_1(h_1)(A) &= 0.5 \\ \beta_1(h_1)(B) &= 0.5 \\ \beta_1(h_2)(L) &= 1 \\ \beta_1(h_2)(R) &= 0\end{aligned}$$

Let's check the equivalence claim. Let  $\sigma_2$  denote a mixed strategy for player 2. Using the mixed strategy  $\sigma_1$ , the probabilities of reaching the outcomes are as follows:

$$\begin{aligned}z_1 &: [\sigma_1(A, L) + \sigma_1(A, R)]\sigma_2(U) = 0.5\sigma_2(U) \\ z_2 &: \sigma_1(A, L)\sigma_2(M) = 0.5\sigma_2(M) \\ z_3 &: \sigma_1(A, R)\sigma_2(M) = 0 \\ z_4 &: \sigma_1(A, L)\sigma_2(D) = 0.5\sigma_2(D) \\ z_5 &: \sigma_1(A, R)\sigma_2(D) = 0 \\ z_6 &: \sigma_1(B, L) + \sigma_1(B, R) = 0.5\end{aligned}$$

The distribution over outcomes using  $\sigma_1$  is then  $(0.5\sigma_2(U), 0.5\sigma_2(M), 0, 0.5\sigma_2(D), 0, 0.5)$ .

Using the behavior strategy  $\beta_1$ , the probabilities of reaching the outcomes are as follows.

$$\begin{aligned}z_1 &: \beta_1(h_1)(A)\sigma_2(U) = 0.5\sigma_2(U) \\ z_2 &: \beta_1(h_1)(A)\sigma_2(M)\beta_1(h_2)(L) = (0.5)\sigma_2(M)(1) = 0.5\sigma_2(M) \\ z_3 &: \beta_1(h_1)(A)\sigma_2(M)\beta_1(h_2)(R) = (0.5)\sigma_2(M)(0) = 0 \\ z_4 &: \beta_1(h_1)(A)\sigma_2(D)\beta_1(h_2)(L) = (0.5)\sigma_2(D)(1) = 0.5\sigma_2(D) \\ z_5 &: \beta_1(h_1)(A)\sigma_2(D)\beta_1(h_2)(R) = (0.5)\sigma_2(D)(0) = 0 \\ z_6 &: \beta_1(h_1)(B) = 0.5\end{aligned}$$

This yields the distribution over outcomes  $(0.5\sigma_2(U), 0.5\sigma_2(M), 0, 0.5\sigma_2(D), 0, 0.5)$  that is the same as the one given by the mixed strategy. Therefore, we have shown that  $\sigma_1$  and  $\beta_1$  are equivalent.

<sup>9</sup>We verify that  $\beta_1(h_1)(A) = 1 - \beta_1(h_1)(B)$ , which is indeed the case.

<sup>10</sup>We again verify that the distribution is valid, which it is because  $\beta_1(h_2)(L) + \beta_1(h_2)(R) = 1$ .



### 2.4.3 A Mixed Strategy Can Generate Many Behavior Strategies

Now let's illustrate the claim that a mixed strategy may generate more than one behavior strategy. Consider the same game and suppose  $\sigma_1(A, L) = \sigma_1(A, R) = 0$ ,  $\sigma_1(B, L) = 0.5$ , and  $\sigma_1(B, R) = 0.5$ . As before, we have  $R_1(h_1) = \{(A, L), (A, R), (B, L), (B, R)\}$ , and  $\pi_1(h_1) = 1$ . Further, we have  $\pi_1(h_1, A) = 0$  (because the mixed strategy assigns probability zero to all pure strategies with  $s_1(h_1) = A$ ), and  $\pi_1(h_1, B) = 1$ . Thus, we get  $\beta_1(h_1)(A) = 0$  and  $\beta_1(h_1)(B) = 1$ .

We now have to specify the probability distribution for the information set  $h_2 = \{(A, M), (A, D)\}$ . Note that  $R_1(h_2) = \{(A, L), (A, R)\}$  and  $\pi_1(h_2) = 0$ . Further,  $\pi_1(h_2, L) = \sigma_1(A, L) = 0$  and  $\pi_1(h_2, R) = \sigma_1(A, R) = 0$ . Hence, we cannot use the conditional formula to define  $\beta_1(h_2)(L)$ . As noted before, in this case we could use any probability distribution, so let's say  $\beta_1(h_2)(L) = x$  and  $\beta_1(h_2)(R) = 1 - x$ , with  $x \in [0, 1]$ . Clearly, there is an infinite number of possible specifications here.

Let's check equivalence. Under the mixed strategy, the probability distribution over outcomes is:

$$\begin{aligned} z_1 : [\sigma_1(A, L) + \sigma_1(A, R)]\sigma_2(U) &= 0 \\ z_2 : \sigma_1(A, L)\sigma_2(M) &= 0 \\ z_3 : \sigma_1(A, R)\sigma_2(M) &= 0 \\ z_4 : \sigma_1(A, L)\sigma_2(D) &= 0 \\ z_5 : \sigma_1(A, R)\sigma_2(D) &= 0 \\ z_6 : \sigma_1(B, L) + \sigma_1(B, R) &= 1. \end{aligned}$$

Under the behavior strategy, the probability distribution is:

$$\begin{aligned} z_1 : \beta_1(h_1)(A)\sigma_2(U) &= 0 \\ z_2 : \beta_1(h_1)(A)\sigma_2(M)\beta_1(h_2)(L) &= (0)\sigma_2(M)x = 0 \\ z_3 : \beta_1(h_1)(A)\sigma_2(M)\beta_1(h_2)(R) &= (0)\sigma_2(M)(1 - x) = 0 \\ z_4 : \beta_1(h_1)(A)\sigma_2(D)\beta_1(h_2)(L) &= (0)\sigma_2(D)x = 0 \\ z_5 : \beta_1(h_1)(A)\sigma_2(D)\beta_1(h_2)(R) &= (0)\sigma_2(D)(1 - x) = 0 \\ z_6 : \beta_1(h_1)(B) &= 1. \end{aligned}$$

That is, the two distributions are the same. Note that this holds for any value of  $x$  we might have chosen. Thus, one mixed strategy can generate more than one behavior strategy. It should be obvious, however, that if the mixed strategy reaches all information sets with positive probability, then it must necessarily generate a unique behavior strategy. Hence, a mixed strategy either generates a unique behavior strategy or else generates an infinite number of behavior strategies.

### 2.4.4 Different Mixed Strategies Can Generate the Same Behavior Strategy

Now let's illustrate the claim that different mixed strategies can generate the same behavioral strategy. Consider the game in Fig. 7 (p. 18). Let  $h_1$  denote the history following action  $U$  by player 1, let  $h_2$  denote the history following  $D$ . Since there are two information sets, with two actions at each, player 2 has four pure strategies:  $(A, C)$ ,  $(A, D)$ ,  $(B, C)$ , and  $(B, D)$ .

Now consider two mixed strategies  $\sigma_2 = (1/4, 1/4, 1/4, 1/4)$  and  $\hat{\sigma}_2 = (1/2, 0, 0, 1/2)$ . Both of these generate the behavior strategy  $\beta_2$ , where  $\beta_2(h_1)(A) = \beta_2(h_1)(B) = 1/2$  and  $\beta_2(h_2)(C) = \beta_2(h_2)(D) = 1/2$ .<sup>11</sup> To see that  $\sigma_2$ ,  $\hat{\sigma}_2$ , and  $\beta_2$  are equivalent, note that they all yield the same distribution over the terminal nodes for any arbitrary mixed strategy for player 1. For example, the probability of reaching  $z_1$  equals

<sup>11</sup>You should verify this. In our notation,  $R_2(h_1) = R_2(h_2) = \{AC, AD, BC, BD\}$ . That is, all strategies for player

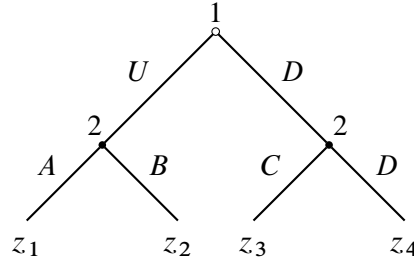


Figure 7: A Game for Kuhn's Theorem, II.

$\sigma_1(U)/2$  regardless of whether we calculate it under  $\sigma_2$ , where it equals  $\sigma_1(U)[\sigma_2(A, C) + \sigma_2(A, D)]$ , or under  $\hat{\sigma}_2$ , where it equals  $\sigma_1(U)[\hat{\sigma}_2(A, C) + \hat{\sigma}_2(A, D)]$ , or under  $\beta_2$ , where it equals  $\sigma_1(U)\beta_2(h_1)(A)$ . As you probably already see, there will be an infinite number of mixed strategies that generate this behavior strategy: All  $\sigma_2$  such that  $\sigma_2(A, C) + \sigma_2(A, D) = 1/2$  and  $\sigma_2(A, C) + \sigma_2(B, C) = 1/2$  will do that.

Although it is important to distinguish between the two types of probabilistic strategies, in practice we shall use behavior strategies throughout the rest of this class. Because it is cumbersome to refer to them as such all the time, whenever we refer to a mixed strategy of an extensive form game, we shall always mean a behavior strategy (unless explicitly noted otherwise). To this end, we shall also retain our  $\sigma$ -notation for mixed strategies: Let  $\sigma_i(a_i|h_i)$  denote the probability with which player  $i$  chooses action  $a_i$  at the information set  $h_i$ .

### 3 Nash Equilibrium in EFG

We already know how to solve strategic form games and we also know how to convert extensive form to strategic form as well. The solution concept we now define ignores the sequential nature of the extensive form and treats strategies as choices to be made by players before all play begins (i.e. just like in strategic games).

**DEFINITION 6.** A **Nash equilibrium of a finite extensive-form game  $\Gamma$**  is a Nash equilibrium of the reduced normal form game  $G$  derived from  $\Gamma$ .

We can do this because the finite extensive form game has a finite strategic form. More generally though, a Nash equilibrium of an extensive form game is a strategy profile  $(s_i^*, s_{-i}^*)$  such that  $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$  for each player  $i$  and all  $s_i \in S_i$ . That is, the definition of Nash equilibrium is the same as for strategic games (but be careful how you specify the strategies here).

Finding the Nash equilibria of extensive form games thus boils down to finding Nash equilibria of their reduced normal form representations. We have already done this with Myerson's card game, reproduced in Fig. 8 (p. 19).

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2 are consistent with these histories. This is trivially true because she has no move to determine which of these histories is reached. We then calculate the probability associated with each history, which, given that all strategies are consistent with it, is simply  $\pi_2(h_1) = \sum_{s_2 \in R_2(h_1)} \sigma_2(s_2) = 1$ . Next, we calculate the probability of taking action  $A$  after  $h_1$ :  $\pi(h_1, A) = \sum_{s_2 \in R_2(h_1) \wedge s_2(h_1)=A} \sigma_2(s_2) = \sigma_2(AC) + \sigma_2(AD) = 0.5$ . Finally, we calculate the behavior strategy  $\beta_2(h_1)(A) = \pi_2(h_1, A)/\pi_2(h_1) = (0.5)/(1) = 0.5$ . We can generate the other strategy in a similar way.

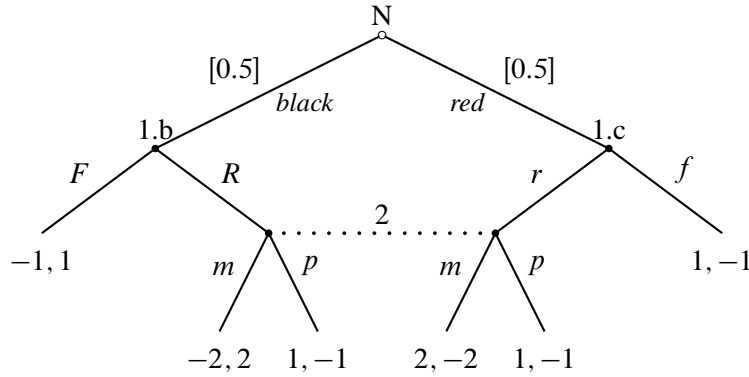


Figure 8: Myerson's Card Game in Extensive Form.

Recall that the mixed strategy Nash equilibrium of this game is:

$$\left\langle \left( \frac{1}{3}[R, r], \frac{2}{3}[F, f] \right), \left( \frac{2}{3}[m], \frac{1}{3}[p] \right) \right\rangle.$$

If we want to express this in terms of behavior strategies, we would need to specify the probability distributions for the information sets. Player 1 has two information sets,  $b$  following the black card, and  $c$  following the red card. The probability distributions are  $(\frac{2}{3}[F], \frac{1}{3}[R])$  at information set  $b$ , and  $(0[f], 1[r])$  at information set  $c$ . In other words, if player 1 sees the black (losing) card, he folds with probability  $\frac{2}{3}$ . If he sees the red (winning) card, he always raises. Player 2's behavior strategy is specified above (she has only one information set).

Because in games of perfect recall mixed and behavior strategies are equivalent (Kuhn's Theorem), we can conclude that a Nash equilibrium in behavior strategies must always exist in these games. This follows directly from Nash's Theorem. Hence, we have the following important result:

**THEOREM 2.** *For any extensive-form game  $\Gamma$  with perfect recall, a Nash equilibrium in behavior strategies exists.* □

### 3.1 The Problem of Counterfactuals

Generally, the first step to solving an extensive-form game is to find all of its Nash equilibria. The theorem tells us at least one such equilibrium will exist. We furthermore know that if we find the Nash equilibria of the reduced normal form representation, we would find all equilibria for the extensive form. Hence, the usual procedure is to convert the extensive-form game to strategic form, and find its equilibria.

Some of these equilibria would have important drawbacks because they ignore the dynamic nature of the extensive-form. This should not be surprising: after all, we obtained the strategic form representation by removing the element of timing of moves completely. Reinhard Selten was the first to argue that some Nash equilibria are "more reasonable" than others in his 1965 article. He used the example in Fig. 9 (p. 20) to motivate the discussion, and so will we.

The strategic form representation has two pure-strategy Nash equilibria,  $\langle D, L \rangle$  and  $\langle U, R \rangle$ .<sup>12</sup> Look closely at the Nash equilibrium  $\langle U, R \rangle$  and what it implies for the extensive form. In the profile  $\langle U, R \rangle$ ,

<sup>12</sup>What about mixed strategies? Suppose player 1 randomizes, in which case player 2's best response is  $L$ . But if this is the case, player 1 would be unwilling to randomize and would choose  $D$  instead. So it cannot be the case that player 1 mixes in equilibrium. What if player 2 mixes? Let  $q$  denote the probability of choosing  $L$ . Player 1's expected payoff from  $U$  is then  $2q + 2(1 - q) = 2$ , and his expected payoff from  $D$  is  $3q$ . He would choose  $U$  if  $2 \geq 3q$ , or  $\frac{2}{3} \geq q$ , otherwise he would choose  $D$ . Player 2 cannot mix with  $1 > q > \frac{2}{3}$  in equilibrium because she has a unique best response to  $D$ . Therefore, she must be mixing with  $0 \leq q \leq \frac{2}{3}$ . For any such  $q$ , player 1 would play  $U$ . So, there is a continuum of mixed-strategy Nash equilibria, where player 1 chooses  $U$ , and player 2 mixes with probability  $q \leq \frac{2}{3}$ . These have the same problem as  $\langle U, R \rangle$ .

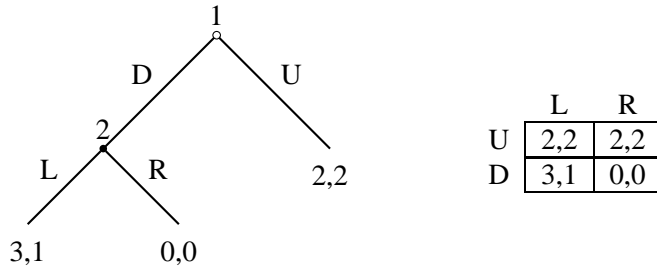


Figure 9: Selten's Example.

player 2's information set is never reached, and she loses nothing by playing  $R$  there. But there is something "wrong" with this equilibrium: *if* player 2's information set is ever reached, then she would be strictly better off by choosing  $L$  instead of  $R$ . In effect, player 2 is threatening player 1 with an action that would not be in her own interest to carry out. Now player 2 does this in order to induce player 1 to choose  $U$  at the initial node thereby yielding her the highest payoff of 2. But this threat is not credible because given the chance, player 2 will always play  $L$ , and therefore this is how player 1 would expect her to play if he chooses  $D$ . Consequently, player 1 would choose  $D$  and player 2 would choose  $L$ , which of course is the other Nash equilibrium  $\langle D, L \rangle$ .

The Nash equilibrium  $\langle U, R \rangle$  is not plausible because it relies on an incredible threat (that is, it relies on an action which would not be in the interest of the player to carry out). In fact, none of the MSNE will be plausible for that very reason either. According to our motivation for studying extensive form games, we are interested in sequencing of moves presumably because players get to reassess their plans of actions in light of past moves by other players (and themselves). That is, nonterminal histories represent points at which such reassessment may occur. The only acceptable solution should be the PSNE  $\langle D, L \rangle$ .

The following definition is very important for the discussion that follows. It helps distinguish between actions that would be taken if the equilibrium strategies are implemented and those that should not.

**DEFINITION 7.** Given any behavior strategy profile  $\sigma$ , and information set is said to be **on the path of play** if, and only if, the information set is reached with positive probability according to  $\sigma$ . If  $\sigma$  is an equilibrium strategy profile, then we refer to the *equilibrium path of play*.

To anticipate a bit of what follows, the problem with the  $\langle U, R \rangle$  solution is that it specifies the incredible action at an information set that is off the equilibrium path of play. Player 2's information set is never reached if player 1 chooses  $U$  (it is a counterfactual). Consequently, Nash equilibrium cannot pin down the optimality of the action at that information set. The problem will not extend to strategy profiles which visit all information sets with positive probability. The reason for this is that if the Nash equilibrium profile reaches all information sets with positive probability, then it will also reach all outcomes with positive probability. But if it does so, the fact that no player can profit by deviating from his Nash strategy implies that there would exist no information set where he would want to deviate. In other words, his actions at all information sets are credible. If, on the other hand, the Nash strategies leave some information sets off the path of play, then the Nash requirement has no bite: whatever the player does at these information sets is "irrelevant" as it cannot affect his payoffs. It is under these circumstances that he may be picking an action that he would not never choose if the information set is actually reached. Notice that unlike  $\langle U, R \rangle$ , the other PSNE  $\langle D, L \rangle$  does reach all information sets with positive probability. In this case, Nash's requirement is sufficient to establish optimality of the strategies everywhere. As we shall see, our solutions will always be Nash equilibria. It's just that not all Nash equilibria will be reasonable.

It is worth emphasizing that the problem is in scenarios that do not arise when the strategies are followed. The problem is especially acute when the cause of these strategies being optimal is incredible

behavior in counterfactual scenarios. The definition of rationality that requires mutual best responses on the path of play only (Nash equilibrium) cannot pin down improbable counterfactuals that rationalize that behavior. We shall, therefore, strengthen the definition of rationality to require that behavior in counterfactual scenarios (contingencies that do not arise when the strategies are followed) is rational in the Nash sense.

### 3.2 Backward Induction

Consider any game of complete and perfect information (that is, a game where all information sets are singletons). Such a game can be solved by **backward induction**, a technique which involves starting from the last stage of the game, determining the last mover's best action at his information set there, and then replacing the information set with the payoffs from the outcome that the optimal action would produce. Continuing in this way, we work upwards through the tree until we reach the first mover's choice at the initial node.

In 1913 Zermelo proved that chess has an optimal solution. He reasoned as follows. Since chess is a finite game (it has quite a few moves, but they are not infinite), this means that it has a set of penultimate nodes. That is, nodes whose immediate successors are terminal nodes. The optimal strategy specifies that the player who can move at each of these nodes chooses the move that yields him the highest payoff (in case of a tie he makes an arbitrary selection). Now, the optimal strategies specify that the player who moves at the nodes whose immediate successors are the penultimate nodes chooses the action which maximizes his payoff over the feasible successors given that the other player moves there in the way we just specified. We continue doing so until we reach the beginning of the tree. When we are done, we will have specified an optimal strategy for each player.

These strategies constitute a Nash equilibrium because each player's strategy is optimal given the other player's strategy. (In fact, these strategies also meet the stronger requirements of subgame perfection, which we shall examine in the next section. Kuhn's paper provides a proof that any finite extensive form game has an equilibrium in pure strategies. It was also in this paper that he distinguished between mixed and behavior strategies for extensive form games.) Hence the following result:

**THEOREM 3 (ZERMELO 1913; KUHN 1953).** *A finite game of perfect information has a pure strategy Nash equilibrium.* □

It is important to realize that this technique ensures Nash behavior in all possible contingencies, including the counterfactuals that do not arise when the optimal strategies are followed. Since Selten's game in Fig. 9 (p. 20) is one of complete and perfect information, we can apply backward induction to find an equilibrium with this feature as well. At her information set, player 2 would choose  $L$ . This reduces player 1's choices between  $D$  (which, given player 2's strategy would yield 3) and  $U$ , which yields 2. Therefore, player 1 would choose  $D$ . The equilibrium with Nash behavior everywhere is  $\langle D, L \rangle$ .

Kuhn's theorem makes no claims about uniqueness of the equilibrium. However, it should be clear that if no player is indifferent between any two outcomes, then the equilibrium will be unique. Note that all equilibria computed with backward induction are Nash equilibria. (The converse, of course, is not true: the whole point of this exercise is to eliminate Nash equilibria that seem implausible.)

### 3.3 A Crisis Escalation Game

Consider the game of complete and perfect information shown in Fig. 10 (p. 22).

What are the Nash equilibria of this game? As usual, we convert this to strategic form, as shown above. (We shall keep the non-reduced version to illustrate a point.) The Nash equilibria in pure strategies are  $\langle (\sim e, a), r \rangle$ ,  $\langle (\sim e, \sim a), r \rangle$ , and  $\langle (e, a), \sim r \rangle$ , and two of them are suspect.

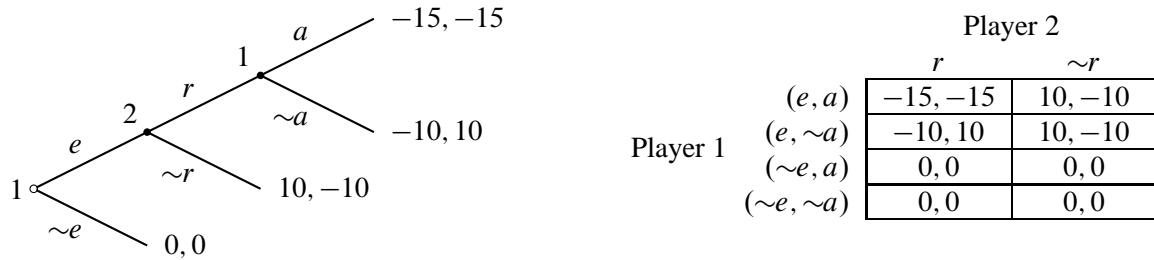


Figure 10: Basic Escalation Game.

The problem with the Nash equilibrium profiles  $\langle (\sim e, a), r \rangle$  and  $\langle (e, a), \sim r \rangle$  is that they leave information sets off the equilibrium path of play and so Nash optimality cannot pin down behavior at sets that are never reached. For example,  $\langle (e, a), \sim r \rangle$  leaves player 1's second information set off the path of play, which causes the strategy to miss the fact that  $a$  is not rational at that set. This incredible threat rationalizes player 2's choice of  $\sim r$  causing her to take an action that leaves this information set off the path. Similarly,  $\langle (\sim e, a), r \rangle$  leaves both player 1's second information set and player 2's information set off the path of play and causes the strategies to miss two problems: player 1's choice of  $a$  is not rational at his second information set, and given that choice, player 2's choice of  $r$  is not rational either! In other words, since the equilibrium path of play does not realize some contingencies, Nash cannot pin down optimal behavior there.

Applying backward induction leaves only  $\langle (\sim e, \sim a), r \rangle$  as the equilibrium, as illustrated in Fig. 11 (p. 22). This eliminates two of the pure-strategy Nash equilibria, and demonstrates why it is extremely important that strategies specify moves even at information sets that would not be reached if the strategy is followed.

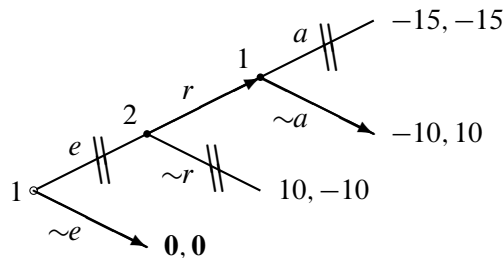


Figure 11: Backward Induction in the Escalation Game.

The only reason why  $\sim e$  is rational at player 1's first information set is because player 2's rational strategy prescribes  $r$ , which in turn is only rational because she expects player 1 to choose  $\sim a$  at his last information set, where this is the rational choice. In other words, the optimality of player 1's initial action depends on the optimality of his action at his second information set. This is precisely why we cannot determine optimality of strategies unless they specify what to do for all information sets. Note that in this case, the second information set is not reached if the strategy  $(\sim e, \sim a)$  is followed, but we still need to know the action there.

### 3.4 The Ultimatum Game

Two players want to split a pie of size  $\pi > 0$ . Player 1 offers a division  $x \in [0, \pi]$  according to which his share is  $x$  and player 2's share is  $\pi - x$ . If player 2 accepts this offer, the pie is divided accordingly. If

player 2 rejects this offer, neither player receives anything. The extensive form of this game is represented in Fig. 12 (p. 23).

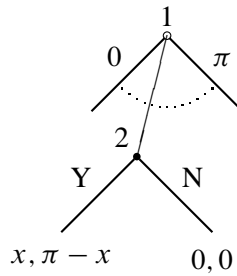


Figure 12: The Ultimatum Game.

In this game, player 1 has a continuum of action available at the initial node, while player 2 has only two actions. (The continuum of actions ranging from offering 0 to offering the entire pie is represented by the dotted curve connecting the two extremes.) When player 1 makes some offer, player 2 can only accept or reject it. There is an infinite number of histories following a history of length 1 (i.e. following a proposal by player 1). Each history is uniquely identified by the proposal,  $x$ . After all histories with  $x < \pi$ , player 2's optimal action is to accept because doing so yields a strictly positive payoff which is higher than 0, which is what she would get by rejecting. After the history  $x = \pi$ , however, player 2 is indifferent between accepting and rejecting. So by backward induction, 2's strategy must be to either accept all offers (including  $x = \pi$ ) or to accept all offers  $x < \pi$  but to reject  $x = \pi$ .

Consider player 1's optimal strategy, which depends on which of player 2's optimal strategies she is supposed to follow. If player 2 accepts all offers, then player 1's optimal offer is  $x = \pi$  because this yields the highest payoff. If player 2 rejects  $x = \pi$  but accepts all other offers, *there is no optimal offer for player 1!* To see this, suppose player 1 offered some  $x < \pi$ , which player 2 accepts. But because player 2 accepts all  $x < \pi$ , player 1 can improve his payoff by offering some  $x'$  such that  $x < x' < \pi$ , which player 2 will also accept but which yields player 1 a strictly better payoff.

Therefore, the ultimatum game has a unique equilibrium obtained by backward induction, in which player 1 offers  $x = \pi$  and player 2 accepts all offers. The outcome is that player 1 gets to keep the entire pie, while player 2's payoff is zero.

This one-sided result comes for two reasons. First, player 2 is not allowed to make any counteroffers. If we relaxed this assumption, as we shall do next, the equilibrium will generally be different. Second, the reason player 1 does not have an optimal proposal when player 2 accepts all offers has to do with him being able to always do a little better by offering to keep slightly more. Because the pie is perfectly divisible, there is nothing to pin the offers. However, making the pie discrete (e.g. by slicing it into  $n$  equal pieces and then bargaining over the number of pieces each player gets to keep) will change this as well.

It is worth noting that sometimes scholars impose a requirement that whenever a player is indifferent between two actions, she chooses one of them. (In this case, it would go something like, "if player 2 is indifferent between accepting some offer and rejecting it, then she accepts it".) This makes it sound like we have introduced a indifference-breaking rule as part of the equilibrium definition. This is not the case! As the discussion above makes clear, there is no equilibrium in which player 2 rejects an offer when indifferent because there is no best response to such a strategy. Thus, it is part of the equilibrium that she accepts it, not some extra requirement.

### 3.5 Finite Horizon Bargaining with Alternating Offers

The Ultimatum Game does not allow player 2 to make counter-offers, which gives all the bargaining leverage to player 1. Let's see what happens if she can reject his proposal in order to make a counter-offer.

Players 1 and 2 are bargaining over the division of a benefit of size  $\pi > 0$  using the alternating offers protocol. Player 1 starts in period 1 by making an offer  $x_1 \in [0, \pi]$ , which player 2 can either accept or reject. If she accepts, the game ends with the split  $(\pi - x_1, x_1)$ . If she rejects, the game moves to period 2, where player 2 makes an offer  $x_2 \in [0, \pi]$  that player 1 can either accept or reject. If player 1 accepts, the game ends with the split  $(x_2, \pi - x_2)$ . If he rejects, the game moves to period 3, where player 1 makes an offer, and so on. The bargaining continues up to  $T$  periods, and ends in  $T + 1$  with both players getting nothing if no agreement has been reached. Players discount periods by a common factor  $\delta \in (0, 1)$ .

We shall consider the game with  $T$  odd, which means that player 1 is the last to make an offer before the game ends (the other case is analogous). Consider the final period  $T$ : this is the Ultimatum Game that we just analyzed. If player 2 rejects  $x_T$ , she will obtain a payoff of 0, and so she is willing to accept any  $x_T > 0$ , and is indifferent if  $x_T = 0$ . In equilibrium, she must accept  $x_T = 0$  (because if she were to reject it with positive probability, player 1 has no best response: any offer better than zero would result in acceptance but each such offer can be improved upon by offering slightly less), and so player 1 would offer  $x_T = 0$ .

Consider now  $T - 1$ : if player 1 rejects  $x_{T-1}$ , he will obtain  $\pi$  in the next period, but because he has to wait for that, his present discounted value of rejection is  $\delta\pi$ . This is now an Ultimatum Game with player 2 making the demand except that player 1 gets a strictly positive payoff if he rejects. Not surprisingly, the logic of the Ultimatum Game tells us that player 2 would extract the entire surplus with her offer. Player 1 would accept any offer that is strictly better than this reservation value and reject anything less than that. In equilibrium, he must accept when indifferent, and since offering anything more than that is not profitable for player 2, she must offer  $x_{T-1} = \delta\pi$ , which will be accepted.

Consider now  $T - 2$ : if player 2 rejects  $x_{T-2}$ , she will obtain  $\pi - \delta\pi = (1 - \delta)\pi$  in the next period, but because she has to wait for that, her present discounted value of rejection is  $\delta(1 - \delta)\pi$ . By the arguments above,  $x_{T-2} = \delta\pi - \delta x_{T-1}$ , which she accepts.

Consider now  $T - 3$ : if player 1 rejects  $x_{T-3}$ , he will obtain  $\pi - x_{T-2}$ , and since he must wait for that, his reservation value is  $\delta(\pi - x_{T-2})$ . By the arguments above, player 2 offers  $x_{T-3} = \delta\pi - \delta x_{T-2}$ , which he accepts.

Consider now  $T - 4$ : player 2's reservation value is  $\delta(\pi - x_{T-3})$ , so player 1's equilibrium offer must be  $x_{T-4} = \delta\pi - \delta x_{T-3}$ .

The pattern is clear: in any period  $t \in \{1, \dots, T - 1\}$ , the offer is defined recursively as

$$x_t = \delta\pi - \delta x_{t+1},$$

with  $x_T = 0$ . You could use difference equations to solve this or you could go about it as follows. We can write the pattern:

$$\begin{aligned} x_{T-1} &= \pi\delta \\ x_{T-2} &= \pi(\delta - \delta^2) \\ x_{T-3} &= \pi(\delta - \delta^2 + \delta^3) \\ x_{T-4} &= \pi(\delta - \delta^2 + \delta^3 - \delta^4) \\ x_{T-5} &= \pi(\delta - \delta^2 + \delta^3 - \delta^4 + \delta^5) \\ &\vdots \end{aligned}$$



$$x_{T-t} = -\pi \cdot \sum_{\tau=1}^t (-\delta)^\tau.$$

There are a couple of ways to solve this. One involves applying the formula for a finite geometric series.<sup>13</sup> The other method is to calculate it using the formulas for finite and infinite sums:<sup>14</sup>

$$x_{T-t} = -\pi \cdot \sum_{\tau=1}^t (-\delta)^\tau = -\pi \left[ \frac{-\delta (1 - (-\delta)^T)}{1 - (-\delta)} \right] = \delta\pi \left[ \frac{1 - (-\delta)^T}{1 + \delta} \right].$$

Since the first-period ( $t = T - 1$ ) offer is accepted, player 1's equilibrium share is:

$$\pi - x_1 = \pi \left[ 1 - \frac{\delta(1 - (-\delta)^{T-1})}{1 + \delta} \right] = \pi \left[ \frac{1 + \delta(-\delta)^{T-1}}{1 + \delta} \right] = \frac{\pi(1 + \delta^T)}{1 + \delta},$$

where the last step follows from the fact that when  $T$  is odd,  $(-\delta)^{T-1} = \delta^{T-1}$ . Player 2's share is, of course, just  $x_1 < \pi - x_1$ .

Thus, we conclude that in the unique SPE players must reach an agreement immediately, and no delay will occur. Note that player 1 has a double advantage: as the *last mover*, he gets to extract the entire surplus in the final period, which then “percolates” to up the game tree with player 2 being forced to make large concessions; and as the *first mover*, he gets to extract the entire surplus from delaying agreement.

What happens to the shares if one increases the number of time periods? It is clear that player 1's share is *decreasing* in  $T$ , and converges to:

$$\lim_{T \rightarrow \infty} \pi - x_1 = \frac{\pi}{1 + \delta},$$

---

<sup>13</sup>The formula is:

$$\sum_{i=1}^n a_i = a \left( \frac{1 - r^n}{1 - r} \right),$$

with a first term  $a_1 = a = -\delta$  and common ratio  $r = -\delta$ . This yields the offer for any arbitrary period:

$$x_{T-t} = \delta\pi \left[ \frac{1 - (-\delta)^t}{1 + \delta} \right].$$

<sup>14</sup>If you cannot recall these formulas, you can derive them as follows. First, you will need to know the sum of the infinite series  $\sum_{t=0}^{\infty} a^t = 1/(1 - a)$ , where  $a \in (-1, 1)$ . This you can get as follows:

$$\sum_{t=0}^{\infty} a^t = 1 + a^1 + a^2 + a^3 + \dots = 1 + a(1 + a^1 + a^2 + \dots) = 1 + a \sum_{t=0}^{\infty} a^t \Rightarrow \sum_{t=0}^{\infty} a^t = \frac{1}{1 - a}.$$

We need a finite sum, which we can express as follows:

$$\sum_{t=0}^T a^t = \sum_{t=0}^{\infty} a^t - \sum_{t=T+1}^{\infty} a^t = \sum_{t=0}^{\infty} a^t - a^{T+1} \sum_{t=0}^{\infty} a^t = (1 - a^{T+1}) \sum_{t=0}^{\infty} a^t = \frac{1 - a^{T+1}}{1 - a}.$$

The indexing in our expression starts at  $t = 1$ , so we obtain:

$$\sum_{t=1}^T a^t = \sum_{t=0}^T a^t - a^0 = \frac{1 - a^{T+1}}{1 - a} - 1 = \frac{a(1 - a^T)}{1 - a}.$$

Letting  $a = -\delta$  yields the result.

which means that longer time horizons make the distribution more equitable by limiting the surplus that player 1 can extract from player 2. Of course, the number of periods will not matter if players do not care about the future: in the extreme, with  $\delta \rightarrow 0$ , player 1 will take everything. However, as players become very patient, player 2 begins to acquire some bargaining leverage, and in the limit, “force” an equal division of the pie:

$$\lim_{\delta \rightarrow 1} \frac{\pi}{1 + \delta} = \frac{\pi}{2}.$$

We shall have occasion to comment on these results when we discuss alternating-offers bargaining without a set time horizon.<sup>15</sup>

#### 4 Subgame-Perfect Equilibrium

If you accept the logic of backward induction, then the following discussion should seem a natural extension. Consider the game in Fig. 13 (p. 26). Here, neither of player 2’s choices is dominated at her second information set: she is better off choosing  $D$  if player 1 plays  $A$  and is better off choosing  $C$  if player 1 plays  $B$ . Hence, we cannot apply backward induction (yet).

However, we can reason in the following way. The game that begins with player 1’s second information set—the one following the history  $(D, R)$ —is a zero-sum simultaneous move game. We have seen similar games, e.g., MATCHING PENNIES. The expected payoffs from the unique mixed strategy Nash equilibrium of this game are  $(0, 0)$ . Therefore, player 2 should only choose  $R$  if she believes that she will be able to outguess player 1 in the simultaneous-move game. In particular, the probability of obtaining 2 should be high enough (in outweighing the probability of obtaining  $-2$ ) that the expected payoff from  $R$  is larger than 1 (the payoff he would get if he played  $L$ ). This can only happen if player 2 believes she can outguess player 1 with a probability of at least  $3/4$ , in which case the expected payoff from  $R$  will be at least  $3/4(2) + 1/4(-2) = 1$ . But, since player 2 knows that player 1 is rational (and therefore just as cunning as she is), it is unreasonable for her to assume that she can outguess player 1 with such high probability. Therefore, player 2 should choose  $L$ , and so player 1 should go  $D$ . The equilibrium obtained by backward induction in the game in Fig. 13 (p. 26), then, is  $((D, 1/2[A]), (L, 1/2[C]))$ .

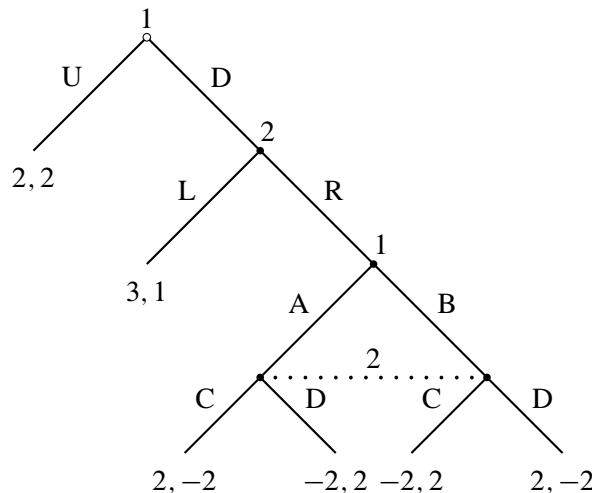


Figure 13: The Fudenberg & Tirole Game.

<sup>15</sup>One must be careful with the limits here. I used nested limits (first on the length of the interaction and then on the discount factor) but one can easily see that the multivariate limit,  $(T, \delta) \rightarrow (\infty, 1)$  is indeterminate – if I were to take the nested limit in the reverse order, I will end up with  $\lim_{\delta \rightarrow 1} \pi - x_1 = \pi$ , which is clearly independent of  $T$ .

This now is the logic of subgame perfection: replace every “proper subgame” of the tree with one of its Nash equilibrium payoffs and perform backward induction on the reduced tree.<sup>16</sup> For the game in Fig. 13 (p. 26), once we replace the subgame that starts at player 1’s second information set with the Nash equilibrium outcome, the game becomes the one in Fig. 9 (p. 20), which we have already analyzed and for which we found that the backward-induction equilibrium is  $\langle D, L \rangle$ .

We were a little vague in the preceding paragraph. Before we formally define what a subgame perfect equilibrium is, we must define what constitutes a “proper subgame.” It really isn’t hard: a proper subgame is any part of a game that can be analyzed as a game itself.

**DEFINITION 8.** A **proper subgame**  $G$  of an extensive-form game  $\Gamma$  consists of a *single* decision node and all its successors in  $\Gamma$  with the property that if  $x' \in G$  and  $x'' \in h(x')$ , then  $x'' \in G$  as well. The payoffs are inherited from the original game.

That is,  $x'$  and  $x''$  are in the same information set in the subgame if and only if they are in the same information set in the original game. The payoffs in the subgame are the same as the payoffs in the original game only restricted to the terminal nodes of the subgame. Note that the word “proper” does not mean strict inclusion as in the term “proper subset.” Any game is always a proper subgame of itself.<sup>17</sup>

Proper subgames are quite easy to identify in a broad class of extensive form games. For example, in games of complete and perfect information, every information set (a singleton) begins a proper subgame (which then extends all the way to the end of the tree of the original game). Each of these subgames represents a situation that can occur in the original game.

On the other hand, splitting information sets in games of imperfect information produces subgames that are not proper because they represent situations that cannot occur in the original game. Consider, for example, the game Fig. 14 (p. 27) and two candidate subgames.

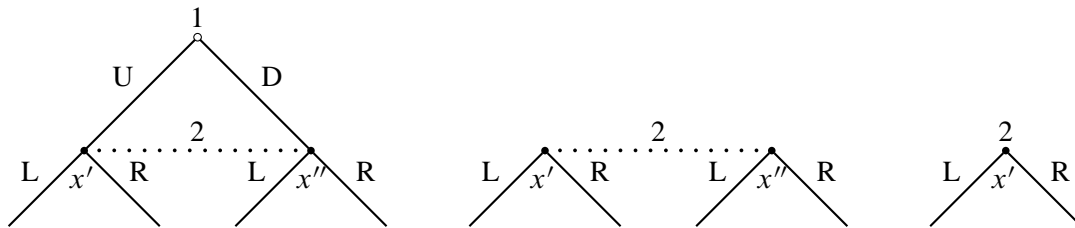


Figure 14: A Game with Two “Improper” Subgames.

The two subgames to the right of the original game are not proper. The first one fails the requirement that a proper subgame begin with a single decision node. The second one fails the requirement that if two decision nodes are in the same information set in the original game, they must also be in the same information set in the proper subgame.

The reasons for these restrictions are intuitive. In the first case, player 2 needs to know the relative probabilities for the decision nodes  $x'$  and  $x''$  but the “game” specification does not provide these probabilities. Therefore, we cannot analyze this situation as a separate game. In the second case, player 2 knows that player 1 did not play  $D$ , and so has more information than in the original game, where he did not know that.

<sup>16</sup>If the game has multiple Nash equilibria, then players must agree on which of them would occur. We shall examine this weakness in the following section.

<sup>17</sup>Rasmusen departs from the convention in his book *Games and Information*, where he defines a proper subgame to mean strict inclusion, and so he excludes the entire game from the set. We shall follow the convention.

To make things a little easier, here are some guidelines for identifying subgames. A subgame (a) always starts with a single decision node, (b) contains all successors to that node, and (c) if it contains a node in an information set, then it contains all nodes in that information set. (Never split information sets.)

Now, given these restrictions, the payoffs conditional on reaching a proper subgame are well defined. We can therefore test whether strategies are a Nash equilibrium in the proper subgame as we normally do. This allows us to state the new solution concept.

**DEFINITION 9.** A behavior strategy profile  $\sigma$  of an extensive form game is a **subgame perfect equilibrium (SPE)** if the restriction of  $\sigma$  to  $G$  is a Nash equilibrium for every proper subgame  $G$ .

You should now see why it was necessary to define the behavior strategies: some proper subgames (e.g. the one in F&T's Game from Fig. 13 (p. 26)) have subgames where the Nash equilibrium is in mixed strategies, which requires that players be able to mix at each information set. (You should at this point go over the difference between mixed and behavior strategies in extensive-form games.)

This now allows us to solve games like the one in Fig. 13 (p. 26). There are three proper subgames: the entire game, the subgame beginning with player 2's information set, and the subgame that includes the simultaneous moves game. We shall work, as we did with backward induction, our way up the tree. The smallest proper subgame has a unique Nash equilibrium in mixed strategies, where each player chooses one of the two available actions with the same probability of .5. Given these strategies, each player's expected payoff from the subgame is 0. This now means that player 2 will choose  $L$  at her information set because doing so nets her a payoffs strictly larger than choosing  $R$  and receiving the expected payoff of the simultaneous-moves subgame. Given that player 2 chooses  $L$  at her information set, player 1's optimal course of action is to go  $D$  at the initial node. So, the subgame perfect equilibrium of this game is  $\langle (D, 1/2), (L, 1/2) \rangle$ .

Let's compare this to the normal and reduced normal forms of this extensive-form game; both of which are shown in Fig. 15 (p. 28).

		Player 2			
		$LC$	$LD$	$RC$	$RD$
Player 1	$UA$	2, 2	2, 2	2, 2	2, 2
	$UB$	2, 2	2, 2	2, 2	2, 2
	$DA$	3, 1	3, 1	2, -2	-2, 2
	$DB$	3, 1	3, 1	-2, 2	2, -2

		$L$	$RC$	$RD$
$U$	2, 2	2, 2	2, 2	
$DA$	3, 1	2, -2	-2, 2	
$DB$	3, 1	-2, 2	2, -2	

Figure 15: The Normal and Reduced Normal Forms of the Game from Fig. 13 (p. 26).

The normal form game on the left has 4 pure strategy Nash equilibria:  $\langle UA, RC \rangle$ ,  $\langle UA, RD \rangle$ ,  $\langle UB, RC \rangle$ , and  $\langle UB, RD \rangle$ . The reduced normal form game has only two:  $\langle U, RC \rangle$  and  $\langle U, RD \rangle$ . None of these are subgame perfect. However, the reduced form also has a Nash equilibrium in mixed strategies,  $\langle \sigma_1^*, \sigma_2^* \rangle$ , in which  $\sigma_1^*(DA) = \sigma_1^*(DB) = 1/2$ , and  $\sigma_1^*(U) = 0$ ; while  $\sigma_2^*(L) = 1$ , and  $\sigma_2^*(RC) = \sigma_2^*(RD) = 0$ . The Nash equilibrium is

$$\langle (0, 1/2, 1/2), (1, 0, 0) \rangle$$

which is precisely the subgame-perfect equilibrium we already found.

At this point you should make sure you can find this mixed strategy Nash equilibrium. Suppose player 2 chooses  $RC$  for sure, then  $DB$  is strictly dominated, so player 1 will not use it. However, this now means  $RD$  strictly dominates  $RC$  for player 2, a contradiction. Suppose now that she chooses  $RD$  for sure. Then  $DA$  is strictly dominated, so player 1 will not use it. But now  $RC$  strictly dominates  $RD$ , a contradiction. Therefore, there is no equilibrium in which player 2 chooses either  $RC$  or  $RD$  for sure.

Suppose player 2 puts positive weight on  $L$  and  $RC$  only. Then,  $DA$  is strictly dominant for player 1. However, 2's best response to  $DA$  is  $RD$ , a contradiction with supposition that she does not play it. Hence, no MSNE in which she plays only  $L$  and  $RC$ . Suppose now that she plays  $L$  and  $RD$  only. Then  $DB$  is strictly dominant, but player 2's best response to this is  $RC$ , a contradiction. Hence, no MSNE in which she plays only  $L$  and  $RD$  either. Suppose next that she plays  $RC$  and  $RD$  only. Then  $U$  is strictly dominant, and since player 2's payoff is the same against  $U$  regardless of the strategy she uses, we have a continuum of MSNE:  $\langle U, \sigma_2(RC) \in (0, 1), \sigma_2(RD) = 1 - \sigma_2(RC) \rangle$ . Suppose next she plays  $L$  for sure. Then player 1 is indifferent between  $DA$  and  $DB$ , each of which strictly dominates  $U$ , so he can mix with  $\sigma_1(DA) \in (0, 1)$  and  $\sigma_1(DB) = 1 - \sigma_1(DA)$ . Since player 2 must not be willing to use any of her other pure strategies, it follows that  $U_2(\sigma_1(DA), RC) \leq 1 \Leftrightarrow \sigma_1(DA) \geq 1/4$ , and  $U_2(\sigma_1(DA), RD) \leq 1 \Leftrightarrow \sigma_1(DA) \leq 3/4$ . Therefore,  $\sigma_1(DA) \in [1/4, 3/4]$  are all admissible mixtures, and we have a continuum of MSNE. The subgame-perfect MSNE is among these: the one with  $\sigma_1(DA) = 1/2$ .<sup>18</sup>

As you can see, we found a lot of MSNE but only one of them is subgame-perfect. This reiterates the point that all SPE are Nash, while not all Nash equilibria are subgame-perfect. Note the different way of specifying the equilibrium in the extensive form and in the reduced normal form.

We can now state a very important result that guarantees that we can find subgame perfect equilibria for a great many games.

**THEOREM 4.** *Every finite extensive game with perfect information has a subgame perfect Nash equilibrium.* □

To prove this theorem, simply apply backward induction to define the optimal strategies for each subgame in the game. The resulting strategy profile is subgame perfect.

Let's revisit our basic escalation game from Fig. 10 (p. 22). It has three subgames, shown in Fig. 16 (p. 29) and labeled I, II, and III. What are the pure-strategy Nash equilibria in all these subgames? We have already found the three equilibria of subgame I:  $\langle (\sim e, a), r \rangle$ ,  $\langle (\sim e, \sim a), r \rangle$ , and  $\langle (e, a), \sim r \rangle$ . The Nash equilibrium of subgame III is trivial:  $\sim a$ . You should verify (e.g. by writing the normal form) that the Nash equilibria of subgame II are:  $\langle a, \sim r \rangle$  and  $\langle \sim a, r \rangle$ .

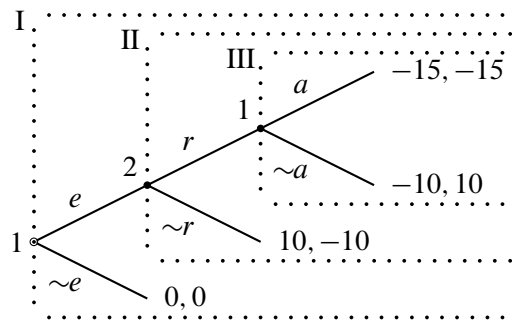


Figure 16: The Subgames of the Basic Escalation Game.

Of the three equilibria in subgame I (the original game), which ones are subgame perfect? That is, in which of these do the strategies constitute Nash equilibria in all subgames? The restriction of the strategies to subgame II shows that no strategy profile that involves anything other than the combinations  $\langle a, \sim r \rangle$  and  $\langle \sim a, r \rangle$  would be subgame perfect. This eliminates the Nash equilibrium profile  $\langle (\sim e, a), r \rangle$  of the original game. Further, the restriction of the strategies to game III demonstrates that no profile that involves player 1 choosing anything other than  $\sim a$  would be subgame perfect either. This eliminates the Nash equilibrium

<sup>18</sup>I leave the final possibility as an exercise: what happens if player 2 puts positive weight on all three of her strategies?

profile  $\langle (e, a), \sim r \rangle$  of the original game. There are no more subgames to check, and therefore all remaining Nash equilibria are subgame perfect. There is only one remaining Nash equilibrium:  $\langle (\sim e, \sim a), r \rangle$  and this is the unique subgame perfect equilibrium. Of course, it is the one we got from our backward induction method as well.

Subgame perfection (and backward induction) eliminates equilibria based upon non-credible threats and/or promises. This is accomplished by requiring that players are rational at every point in the game where they must take action. That is, their strategies must be optimal at every information set, which is a much stronger requirement than the one for Nash equilibrium, which only demands rationality at the first information set.

Note that nowhere in our definition of extensive form games did we restrict either the number of actions available to players at their decision nodes nor the number of decision nodes. For example, a player may have a continuum of actions or some terminal history may be infinitely long. If a player has a continuum of action at any decision node, then there is an infinite number of terminal histories as well. We must distinguish between games that exhibit some finiteness from those that are infinite.

If the length of the longest terminal history is finite, then the game has **finite horizon**. If the game has finite horizon and finitely many terminal histories, then the game is **finite**. Backward induction only works for games that are finite. Subgame perfection works fine for infinite games.

#### 4.1 The Dollar Auction

Let's now play the following game. I have \$1 that I want to auction off using the following procedure. Each student can bid at any time, in 10 cent increments. When no one wants to bid further, the auction ends and the dollar goes to the highest bidder. Both the highest bidder and the second highest bidder pay their bids to me. Each of you has \$3.00 to bid with and you cannot bid more than that.

#### [ What happened? ]

Let's analyze this situation by applying backward induction. Consider a game with 2 players, and assume that it is not worth spending a dollar in order to win a dollar. Because of the budget constraint, whoever bids \$3.00 will win the auction. When would a player ever want to bid \$3.00? Clearly, the only reason to do so would be if that player is attempting to avoid a loss from an existing bid. Let  $x > 0$  be that player last bid that he would have to pay if he loses the auction. He would bid \$3.00 if

$$\$1 - \$3.00 > -x \quad \Rightarrow \quad x > \$2.00 \quad \Rightarrow \quad x \geq \$2.10,$$

where the last step follows from the assumption that players can only bid in \$0.10 increments. That is, if a player ever bids \$2.10 or more, then he must be willing to go all the way up to \$3.00 and win the auction. The reason is simple: not bidding at this point would entail a loss of \$2.10, whereas bidding at the maximum would entail a loss of \$2.00 only. Thus, whoever bids \$2.10 first has a credible threat to escalate the auction to \$3.00 and win it. This means that the other player has no incentive to attempt to outbid him. In effect, the \$2.10 bid is "equivalent" to the winning \$3.00 bid in that it also wins the auction immediately.

Why would anyone bid \$2.10 though? Again, since doing so and winning entails a certain loss, it must be that not doing so would incur an even bigger loss. In other words, the player willing to escalate to \$2.10 must have made a previous bid that they would lose unless they escalate. Let  $y > 0$  be that player's existing bid. He would bid \$2.10 only if

$$\$1 - \$2.10 > -y \quad \Rightarrow \quad y > \$1.10 \quad \Rightarrow \quad y \geq \$1.20,$$

where we used the \$0.10 increment rule again. So, if some player ever bids \$1.20 or more, then he must be willing to go all the way to \$2.10 to win the auction, where, one should recall, the bidding would have to end for sure because this player has a credible threat to go all the way to \$3.00 in order to win. This means that the other player has no incentive to attempt to outbid him, which would end the auction immediately. In effect, the \$1.20 bid is “equivalent” to the winning \$2.10 and \$3.00 bids in that it also wins the auction immediately.

Why would anyone bid \$1.20 though. Since this still entails a loss even in winning the auction, it must be that the player’s previous bid is positive and he does not want to lose it. Let  $z > 0$  denote that bid. The player would bid \$1.20 only if

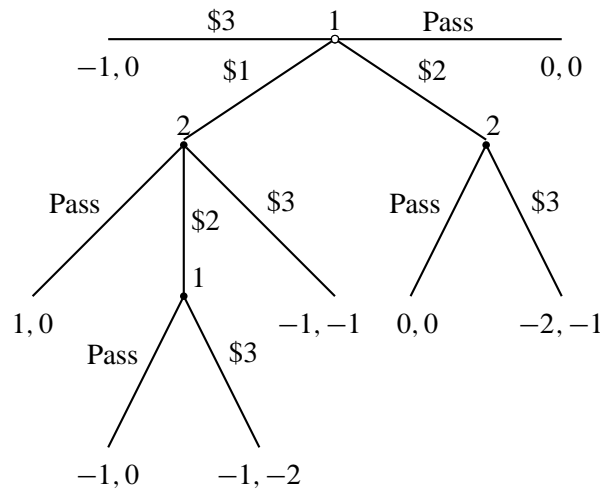
$$\$1 - \$1.20 > -z \quad \Rightarrow \quad z > \$0.20 \quad \Rightarrow \quad z \geq \$0.30,$$

where the increment rule came into play again. Whoever bids \$0.30 or more must be willing to go all the way to \$1.20 to win the auction. Since that player has a credible threat to continue up to \$3, the other player has no incentive to attempt to outbid him, and so the auction must end immediately. In effect, the \$0.30 bid is “equivalent” to the winning \$1.20, \$2.10, and \$3.00 bids in that it also wins the auction immediately.

If someone bids \$0.30, then nobody with a smaller bid has an incentive to challenge him, and knowing this no such bids should be made. Thus, in the SPE, the player that moves first should bid \$0.30, and the auction would end, giving him a profit of \$0.70 and allowing the other player not to suffer any losses. (The auctioneer would lose here.)

How does that correspond to our outcome? I bet (pun intended) not too well. I usually auction off the dollar for a profit of about \$2 with grad students, and more than double that with undergrads.

Before pondering why, let us look at an extensive-form representation of a simpler variant, where two players have \$3.00 each but they can only bid in dollar increments in an auction for \$2.00.



Assume, as before, that if a player is indifferent between bidding and passing, they pass. We begin with the longest terminal history, (\$1, \$2), and consider 1’s decision there. Since 1 is indifferent between Passing and bidding \$3, player 1 would pass. This means that player 2 would be indifferent between bidding \$2 and passing at the decision node following history (\$1), and so she would pass. The subgame beginning at the information set (\$1) has a unique SPE in pure strategies: (Pass, Pass).

At the decision node following history (\$2), player 2’s unique optimal action is to pass, and so the subgame perfect equilibrium there is (Pass). Therefore, player 2’s strategy must specify Pass for this decision node in any SPE.

Consider now player 1's initial decision. Since the players' strategies are such that they play the (Pass, Pass) SPE in the subgame after history (\$1), then player 1 does best by bidding \$1 at the outset. Therefore,  $(\$1, \text{Pass}), (\text{Pass}, \text{Pass})$  is the SPE of the game under the indifference rule. The outcome is that player 1 bids \$1 and player 2 passes. This corresponds closely to the outcome in our discussion above.

There is a general formula, due to Barry O'Neill, that you can use to calculate the optimal size of the first bid, which depends on the amount of money available to each bidder, the size of the award, and the amount of bid increments. Let the bidding increment be represented by one unit (so the unit in our example is a dime). If each player has  $b$  units available for bidding, and the award is  $v$  units, the optimal bid is  $((b - 1) \bmod (v - 1)) + 1$ . In our example, this translates to  $(30 - 1) \bmod (10 - 1) + 1 = 3$  units, which equals 30 cents, just as we found.

It is interesting to note that the size of the optimal bid is very sensitive to the amount each player has available for bidding. If each player has \$2.80 instead of \$3.00, then the optimal bid is  $(28 - 1) \bmod (10 - 1) + 1 = 1$ , or just 10 cents. If, however, each has \$2.70, then the optimal bid is  $(27 - 1) \bmod (10 - 1) + 1 = 9$ , or 90 cents.

The Dollar Auction was first described by Martin Shubik who reported regular gains from playing the game in large undergraduate classes. The game is a very useful thought experiment about escalation. At the outset, both players are trying to win the prize by costly escalation, but at some point the escalation acquires momentum of its own and players continue paying costs to avoid paying the larger costs of capitulating. The requirement that both highest bidders pay the cost captures the idea of escalation.

There is just one problem with using the game to explain escalation: in the SPE no escalation occurs! Whereas the cost-avoiding logic does establish the credibility of threats off the path of play to continue until the ultimate end (which in itself "punctuates" the play by establishing thresholds where particular bids must end the game immediately), this very credibility induces player to avoid incurring any unnecessary costs in the first place: the player who makes the appropriate initial bid wins instantly, and benefits from doing so.

The no-escalation SPE stands in stark contrast with experimental results where escalation is very common, and where it often involves costs that well exceed the benefit of winning. One could argue that players in these experiments do not understand the game at the outset, and by the time they figure it out, they are locked in the cost-avoidance phase of the game. But the problem there is that usually even after only two players are left in that phase, they keep bidding instead of one of them immediately hitting one of the thresholds and winning. (This is why I had to restrict the budgets to \$3: I once tried the game with \$10 budgets and the players did not stop until they almost exhausted them.) Players do understand the cost-avoidance logic — what they do not seem to quite settle on is the credibility of the threat to continue the escalation once a threshold bid is made.

To me this suggests that in these experiments players might not be playing the game as specified but a variant of it that involves incomplete information about the abilities or risk-propensities of the other players. The problem is not that the players are irrational in the Nash sense but that the experimenter has not been able to control the experiment. The intuition is that the SPE result is predicated on the threats being credible (to all involved), and the cost-avoidance logic establishes the credibility to go all out at certain bid levels. If there is a small probability that a bid just above the particular threshold would win the auction (e.g., because the other player makes a mistake and quits or because she is risk-averse and quits instead of escalating as she should or because there is an exogenous probability of being declared the winner), then the magnitude of the threat would matter. In fact, if the losses incurred with the previous bid are not very high, then one might be tempted to "cross" the threshold, causing the game to escalate to the next one threshold, where a similar problem might re-occur. Since the costs become larger as the game nears the end, the incentive to try one's luck gets weaker, and so the game becomes increasingly likely



to end at the threshold. If escalation has this property (and many real-life situations probably do have it), then one could explain escalation with the model suitably adjusted.<sup>19</sup>

## 4.2 Sophisticated Voting and Agenda Control

Suppose there are three players,  $J = \{1, 2, 3\}$ , who must choose one from three alternatives  $\mathcal{X} = \{x, y, z\}$ . Their preferences are as follows:

- Player 1:  $x \succ y \succ z$ ;
- Player 2:  $y \succ z \succ x$ ;
- Player 3:  $z \succ x \succ y$ .

They must make their choice through majority rule voting in a two-stage process. They first vote on two of the alternatives and the winner is then pitted against the remaining alternative in the second round. Players cast their votes simultaneously in each of the two rounds. Suppose player 2 controls the agenda—that is, she can decide which two alternatives are to be voted on in the first round. What is her choice?

We need to find the SPE. Clearly, player 2 will set an agenda that ensures that  $y$  is the winner if that's possible. We have to consider the three possible situations, depending on which two alternatives are selected in the first round. To find the subgame-perfect equilibrium, we have to ensure that the strategies are optimal in all subgames. There are three generic subgames that begin with the second round: depending on the winning alternative in the first round, the second round can involve a vote on  $(x, y)$ , or  $(x, z)$ , or  $(y, z)$ . So let's analyze each of these subgames.

Note first that because there are only two alternatives and three players, it follows that in any PSNE, at least two players must vote for the same alternative. Observe now that casting a *sincere* vote—that is, voting for the preferred alternative—is weakly dominant for each player. If the other two players vote for different alternatives, then the third player's vote is decisive, and it is strictly better to cast it sincerely. If, on the other hand, the two other players vote for the same alternative, then the third player's vote cannot change the outcome. Therefore, there are two possible PSNE in these subgames: either all players vote sincerely or they all vote for the same alternative. Technically, this means that when  $x$  is pitted against  $y$ , it is possible to get  $y$  to win: the strategy profile in which all players vote for  $y$  is a Nash equilibrium. However, this PSNE requires two of the players to vote against their preferred alternative and *expect that they do so*, which seems highly implausible. In this instance, I would rule out the PSNE involving weakly dominated strategies. Using only weakly dominant strategies then yields a unique PSNE with sincere voting for each of the subgames, as follows:

- if  $(x, y)$ , then  $x$  wins (1 and 3 vote  $x$  and 2 votes  $y$ );
- if  $(x, z)$ , then  $z$  wins (2 and 3 vote  $z$  and 1 votes  $x$ );
- if  $(y, z)$ , then  $y$  wins (1 and 2 vote  $y$  and 3 votes  $z$ ).<sup>20</sup>

In other words, we know that in SPE the second round will involve sincere voting and produce a winner accordingly.

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<sup>19</sup>This, in fact, is the essence of the model that Bahar Leventoglu and I proposed to explain why wars might not be settled immediately despite both sides having complete information — in our case, if a player escalates unexpectedly, there is a small exogenous chance that the other player would collapse.

<sup>20</sup>Of course, there are also the PSNE in which two players vote sincerely and the third, whose vote is irrelevant, also votes for the same alternative. For instance, in  $(x, y)$ , there is PSNE in which all players vote for  $x$ . Since player 3's choice cannot affect the outcome, he might as well vote insincerely.

Going back to our original question, how is player 2 to set the agenda? Clearly, if  $y$  is ever to emerge as the winner given that the 2nd round will involve sincere voting, it will have to be pitted there against  $z$  because against  $x$  it will lose. Since  $z$  defeats  $x$  in sincere voting, then perhaps choosing  $(x, z)$  as the first round agenda would work?

The answer is that it will not. Suppose player 2 set the agenda with  $(x, z)$  in the first round and everyone voted sincerely. Then the winner would be  $z$ , and in the second round the winner would be  $y$ . But players can anticipate this outcome. In particular, player 3 knows that the winner of the first comparison would go on to compete with  $y$  and if  $y$  prevails in the 2nd round, he will get his worst possible alternative. Since  $y$  will beat  $z$  with sincere voting, this means that he really does not want  $z$  to win in the first round. If votes are cast sincerely in the first round, then player 1 is voting for  $x$  while players 2 and 3 are voting for  $z$ . However, if player 3 deviated and cast a *sophisticated* vote for  $x$  instead, then  $x$  will win the first round, and in the second round the sincere vote on  $(x, y)$  would leave  $x$  as the winner. Although the sophisticated vote does not enable player 3 to get his most preferred alternative, it does enable him to avoid the worst possible one.<sup>21</sup>

This now means that whatever player 2 chooses, her agenda has to be invulnerable to sophisticated voting. Well, as they say, if you can't beat them, join them: player 2 will exploit the sophistication of the players by setting the agenda for the first round to  $(x, y)$ . Observe that with sincere voting,  $x$  would defeat  $y$ . However, this would pit  $x$  against  $z$  in the 2nd round, in which case  $z$  will prevail. Player 1 can foresee this and since  $z$  is his worst possible alternative, he will cast a sophisticated vote for  $y$  against her preference for  $x$  over  $y$ . Doing so would ensure that  $y$  will go on to the 2nd round and defeat  $z$ , which gives him the second-best outcome. Of course, our devious player 2 can now enjoy her most preferred alternative. Therefore, the profile

$$\langle (y, xy), (y, z, y), (x, zz) \rangle$$

is a subgame-perfect equilibrium when  $(x, y)$  is the pair in the first round. The strategies are specified as a triple over the  $(x, y)$  choice in the first round, and then the  $(x, z)$  and  $(y, z)$  possible subgames in the second round. In this SPE player 1 is casting a sophisticated vote (note that player 2's manipulation pays off even though she votes sincerely). Alternative  $y$  defeats  $x$ , and then goes on to defeat  $z$  in a sincere vote in the 2nd round. Since this SPE yields player 2 her most preferred alternative, the overall SPE of the game involves her setting the agenda such that  $(x, y)$  are the two competing alternatives in the first round.

We know from McKelvey's Chaos Theorem that if players vote sincerely using majority rule to select winners in pairwise comparisons, then any outcome is possible provided no equilibrium position exists. (That is, for any two alternatives, one can always find an agenda that guarantees that one beats the other.) With sophisticated voting, this chaos is a bit reduced: for any two alternatives, there will be an agenda that guarantees that one defeats the other only if the winner can also beat the loser in a majority vote with sincere voting or there is a third alternative that can defeat the loser and itself be defeated by the winner in a majority vote with sincere voting (this is due to Shepsle and Weingast). In our situation  $y$  can beat  $z$  on its own with sincere voting, and  $y$  can beat  $x$  through  $z$  because  $z$  can defeat  $x$  in sincere voting and  $y$  in turn defeats  $z$ . Hence, there is an agenda that ensures  $y$  is reachable.<sup>22</sup>

Agenda-setting gives player 2 the ability to impose her most preferred outcome and there is nothing (in this instance) that the others can do. For instance, player 1 and player 3 cannot collude to defeat her obvious intent. To see this, suppose player 3 proposed a deal to player 1: if player 1 would vote sincerely for  $x$  in the first round, then player 3 would reward him by voting for  $x$  on the 2nd round. Since  $x$  will

<sup>21</sup>Here, as before, there are equilibria in which all three players vote for the same alternative and two of them vote against their preferences. For instance, in  $(x, z)$  this would require them all to vote for  $x$ . Coupling this with any PSNE in the 2nd round will yield an SPE, but the solution is implausible for the same reasons we discussed already.

<sup>22</sup>In our game, each of the three outcome is possible with an appropriate agenda. Sophisticated voting does not reduce the chaos.

then beat  $y$  in the first round, player 3's insincere vote in the second round would ensure that  $x$  will defeat  $z$  as well. This would benefit both players: player 1 would get his most preferred outcome and player 3 would avoid the worst outcome  $y$  and get his second-best. Unfortunately (for player 3), he cannot make a credible promise to cast an insincere vote. If  $x$  defeats  $y$  in the first round, then player 3 can get his most preferred outcome by voting sincerely in  $(x, z)$  in the second round. Therefore, he would renege on his pledge, so player 1 has no incentive to believe him. But since this renegeing would saddle player 1 with his worst outcome, player 1 would strictly prefer to cast his sophisticated vote in the first round even though he is perfectly aware of how player 2 has manipulated the agenda to her advantage. The inability to make credible promises, like the inability to make credible threats, can seriously hurt players. In this instance, player 3 gets the worst of it.

### 4.3 The Holdup Game

Credible commitment issues crop up in various settings. Consider the following three-stage game. Before playing the Ultimatum Game from the previous section, player 2 can determine the size of the pie by exerting a small effort,  $e_S > 0$  resulting in a small pie of size  $\pi_S$ , or a large effort,  $e_L > e_S$ , resulting in a larger pie of size  $\pi_L > \pi_S$ . Since player 2 hates exerting any efforts, her payoff from obtaining a share of size  $x$  is  $x - e$ , where  $e$  is the amount of effort expended. The extensive form of this game is presented in Fig. 17 (p. 35).

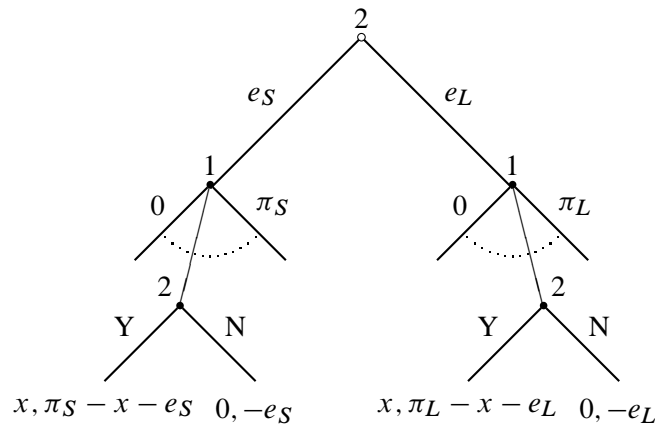


Figure 17: The Holdup Game.

We have already analyzed the Ultimatum Game, so each subgame that follows player 2's effort has a unique SPE where player 1 proposes  $x = \pi$  and player 2 accepts all offers (note that the difference between this version and the one we saw above is that player 2 gets a strictly negative payoff if she rejects an offer instead of 0). So, in the subgame following  $e_S$ , player 1 offers  $\pi_S$  and in the subgame following  $e_L$  he offers  $\pi_L$ . In both cases player 2 accepts these proposals, resulting in payoffs of  $-e_S$  and  $-e_L$  respectively. Given these SPE strategies, player 2's optimal action at the initial node is to expend little effort, or  $e_S$  because doing so yields a strictly better payoff.

We conclude that the SPE of the Holdup Game is as follows. Player 1's strategy is  $(\pi_S, \pi_L)$  and player 2's strategy is  $(e_S, Y, Y)$ , where  $Y$  means "accept all offers." The outcome of the game is that player 2 invests little effort,  $e_S$ , and player 1 obtains the entire small pie  $\pi_S$ .

Note that this equilibrium does not depend on the values of  $e_S, e_L, \pi_S, \pi_L$  as long as  $e_S < e_L$ . Even if  $\pi_L$  is much larger than  $\pi_S$  and  $e_L$  is only slightly higher than  $e_S$ , player 2 would still exert little effort in SPE although it would be better for both players if player 2 exerted  $e_L$  (remember, only *slightly* larger than  $e_S$ ) and obtained a slice of the larger pie. The problem is that player 1 cannot credibly promise to give that

slice the player 2. Once player 2 expends the effort, she can be “held up” for the entire pie by player 1.

This result holds for similar games where the bargaining procedure yields a more equitable distribution. If player 2 must expend more effort to generate a larger pie and if the procedure is such that some of this surplus pie goes to the other player, then for some values of player 2’s cost of exerting this effort, she would strictly prefer to exert little effort. Although there are many outcomes where both players would be strictly better off if player 2 exerted more effort, these cannot be sustained in equilibrium because of player 1’s incentives. In the example above, player 1 would have liked to be able to commit credibly to offering some of the extra pie to induce player 2 to exert the larger effort. Just like the problem with non-credible threats, the problem of non-credible promises means that this cannot happen in subgame perfect equilibrium.

#### 4.4 A Two-Stage Game with Several Static Equilibria

Promises about future behavior can be credible and useful when the behavior involve equilibrium play. We now look at an example of this that is also a useful introduction to some ideas that we shall develop at length when we turn to repeated games next.

Consider the game corresponding to two repetitions of the symmetric normal form game depicted in Fig. 18 (p. 36). In the first stage of the game, the two players simultaneously choose among their actions, observe the outcome, and then in the second stage play the static game again. The payoffs are simply the discounted average from the payoffs in each stage. That is, let  $p_i^1$  represent player  $i$ ’s payoff at stage 1 and  $p_i^2$  represent his payoff at stage 2. Then player  $i$ ’s payoff from the multi-stage game is  $u_i = p_i^1 + \delta p_i^2$ , where  $\delta \in (0, 1)$  is the discount factor.

		Player 2		
		A	B	C
Player 1	A	0, 0	3, 4	6, 0
	B	4, 3	0, 0	0, 0
	C	0, 6	0, 0	5, 5

Figure 18: The Static Period Game.

If the game in Fig. 18 (p. 36) is played once, there are three Nash equilibria, two asymmetric ones in pure strategies:  $\langle B, A \rangle$ ,  $\langle A, B \rangle$ , and one symmetric in mixed strategies with  $\sigma(A) = 3/7$ , and  $\sigma(B) = 4/7$ .

How do we find the MSNE? You should notice that the PSNE do not involve  $C$  for any of the players, so perhaps they would not play this in MSNE either? To check our intuition, suppose some player chooses  $\sigma_i^*(C) > 0$  in MSNE. Since  $A$  weakly dominates  $C$  for  $i$ , the only reason  $C$  could be played with positive probability is that  $A$ ’s advantage is not realized; that is, the other player cannot be playing either  $B$  or  $C$  with positive probability. Thus, the other player must be choosing  $A$  with certainty or else player  $i$  would never choose  $C$  since it would be strictly dominated by  $A$ , or  $\sigma_i^*(C) > 0 \Rightarrow \sigma_{-i}^*(A) = 1$ . But if  $-i$  is choosing  $A$  with certainty, then player  $i$  has a unique best response, which is to play  $B$ , or  $\sigma_i^*(B) = 1 \Rightarrow \sigma_i^*(C) = 0$ , a contradiction. Since the game is symmetric, we can conclude that  $\sigma_i^*(C) = 0$  for every player  $i$  in equilibrium. This means that we can find the MSNE by considering the  $2 \times 2$  game in Fig. 19 (p. 36).

		Player 2	
		A	B
Player 1	A	0, 0	3, 4
	B	4, 3	0, 0

Figure 19: The Static Period Game after Some Equilibrium Logic..

This is now very easy to deal with. Since player 1 is willing to mix, it follows that  $3\sigma_2(B) = 4\sigma_2(A)$  and since  $\sigma_1(B) = 1 - \sigma_2(A)$ , this gives us  $\sigma_2(A) = 3/7$ . Analogously, we obtain  $\sigma_1(A) = 3/7$  and  $\sigma_1(B) = 4/7$ . The last thing we need to do is check that the players will not want to use  $C$  given the mixtures (we already know this from the argument above, but it does not hurt to recall the MSNE requirement). It suffices to check for player 1: if he plays  $C$ , his payoff will be 0 given player 2's strategy of playing only  $A$  and  $B$  with positive probability, which is strictly worse than the expected payoff from either  $A$  or  $B$ . Hence, we do have our MSNE indeed. The payoffs in the three equilibria are  $(4, 3)$ ,  $(3, 4)$ , and  $(12/7, 12/7)$  respectively.

The efficient payoff  $(5, 5)$  is not attainable in equilibrium with positive probability if the game is played once.<sup>23</sup> It is easy to see nine SPE in the 2-period game: the strategy profiles that simply specify strategies that are Nash equilibrium of the stage game unconditionally. That is, take one of the three Nash equilibria for all the subgames in the 2nd period (this is what players would do no matter what they did in the first period). Then the first period's behavior does not affect the 2nd period by construction, and so one only need to worry about profitable deviations from the strategies in the 1st period. But playing any of the Nash equilibria there guarantees that no such profitable deviation would exist. Since there are 3 Nash equilibria, there are  $3 \times 3 = 9$  unconditional SPE that can be constructed in this way. In none of these is the efficient payoff  $(5, 5)$  obtained with positive probability in any of the periods.

Since it is not possible to obtain this payoff in the 2nd period (as no Nash equilibrium would permit it), the only possibility is that it obtains in the 1st period. That is, the strategies for the game would specify non-Nash play in the first period. By definition, this means that each player has a profitable deviation for the period, so to remove the incentive to deviate, there must be some negative consequences in the 2nd period when he does deviate, and perhaps positive rewards if he does not. Since all SPE require Nash behavior in the 2nd period, the different rewards and costs must be obtainable in a Nash equilibrium of the stage game. Looking at the 3 Nash equilibria, it is clear that the PSNE  $\langle A, B \rangle$  and  $\langle B, A \rangle$  are *rewards* since they yield relatively high payoffs for the players, whereas the MSNE is a *punishment* since it yields relatively low payoffs.

Let us condition "cooperative" behavior in the first period with a system of rewards and punishments in the second. Consider the following strategy profile for the two-stage game:

- Player 1: play  $C$  at the first stage. If the outcome is  $(C, C)$ , play  $B$  at the second stage, otherwise play  $\sigma_1(A) = 3/7, \sigma_1(B) = 4/7$  at the second stage;
- Player 2: play  $C$  at the first stage. If the outcome is  $(C, C)$ , play  $A$  at the second stage, otherwise play  $\sigma_1(A) = 3/7, \sigma_2(B) = 4/7$  at the second stage.

That is, playing  $(C, C)$  in the first period is rewarded with  $(B, A)$  in the second period, and deviation is punished with the MSNE. Is it subgame perfect? Since the strategies at the second stage specify playing Nash equilibrium profiles for all possible second stages, the strategies are optimal there. At the first stage players can deviate and increase their payoffs by 1 from 5 to 6 (either player can choose  $A$ ). However, doing so results in playing the mixed strategy Nash equilibrium at the second stage, which lowers their payoffs to  $12/7$  from 4 for player 1 and from 3 for player 2. Thus, player 1 will not deviate if:

$$\begin{aligned} 6 + \delta(12/7) &\leq 5 + \delta(4) \\ 1 &\leq \delta(4 - 12/7) \\ \delta &\geq 7/16 \end{aligned}$$

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<sup>23</sup>An outcome is *efficient* if it is not possible to make some player better off without making the other one worse off. The outcomes with payoffs  $(0, 0)$  are all inefficient, as are the outcomes with payoffs  $(4, 3)$  and  $(3, 4)$ . Efficiency does not imply equity, however: the outcomes  $(6, 0)$  and  $(0, 6)$  are also efficient.

Similarly, player 2 will not deviate if:

$$\begin{aligned} 6 + \delta (12/7) &\leq 5 + \delta(3) \\ 1 &\leq \delta (3 - 12/7) \\ \delta &\geq 7/9 \end{aligned}$$

We conclude that the strategy profile specified above is a subgame perfect equilibrium if  $\delta \geq 7/9$ . Another cooperative SPE exists with the  $(A, B)$  being the reward in the 2nd period. Both work in the same way: players attain the non-Nash efficient outcome at stage 1 by threatening to revert to the worst possible Nash equilibrium at stage 2. This technique will be very useful when analyzing infinitely repeated games, where we shall see analogous results.

#### 4.5 A Problem with Commitment

Let us look at yet another manifestation of the commitment problem (the players being unable to make credible promises about their own future behavior).

Two players are bargaining over the division of a benefit of size 1. Consider first a single-period interaction. If both players agree to some division  $(x, 1 - x)$  with  $x \in [0, 1]$  being player 1's share, it is implemented immediately and players obtain instantaneous per-period payoffs

$$u_1(x) = x \quad \text{and} \quad u_2(x) = 1 - x.$$

Each player can also choose to impose a solution by force. Using force is costly, with each player  $i$  paying  $c_i > 0$ . Moreover, the outcome is uncertain: player 1 wins with probability  $p \in (0, 1)$  and loses with probability  $1 - p$ . Using force is a winner-take-all costly lottery, so the expected payoff is:

$$\begin{aligned} w_1 &= pu_1(1) + (1 - p)u_1(0) - c_1 = p - c_1 \\ w_2 &= pu_2(1) + (1 - p)u_2(0) - c_2 = 1 - p - c_2. \end{aligned}$$

We will now obtain some very general results that do not depend on the bargaining protocol. If using force is always an option for each player, then neither would agree to any peaceful deal that is worse than the expected payoff from war. Thus, peace requires that  $u_i(x) \geq w_i$  for each  $i \in \{1, 2\}$ . It will always be possible to find such a deal whenever the sum of what players get in peace is at least as good as their combined war expectations:

$$u_1(x) + u_2(x) \geq w_1 + w_2.$$

This condition is always satisfied in this model:

$$u_1(x) + u_2(x) = x + 1 - x = 1 > p - c_1 + 1 - p - c_2 = 1 - (c_1 + c_2) = w_1 + w_2.$$

In other words, there exist deals that can satisfy both players' fighting expectations and make at least one of them strictly better off with peace (all but two such deals make both of them better off).

Fig. 20 (p. 39) illustrates this graphically. Since players are risk-neutral, they will accept any deal that gives them with certainty at least as much as their expected (risky) payoff from war. Thus, player 1 would accept any division  $x \geq p - c_1$ , whereas player 2 would accept any division  $1 - x \geq 1 - p - c_2$ , or  $x \leq p + c_2$ . Clearly, then all divisions  $x \in [p - c_1, p + c_2]$  are preferable to war for both players, so they are mutually acceptable. This is called the *bargaining range*, and it always exists as long as  $c_i > 0$ . The size of the bargaining range,  $c_1 + c_2$ , is called the *bargaining surplus*, and it represents what there is to be divided peacefully after satisfying both players' minimum demands (their war payoffs). Since

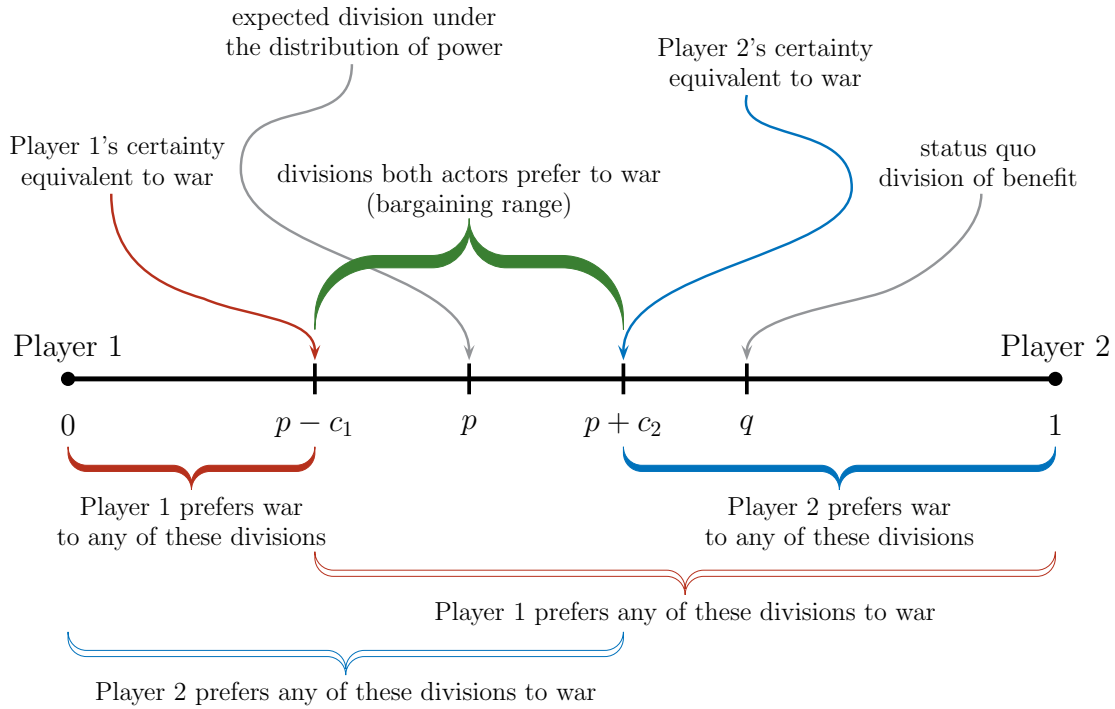


Figure 20: Mutually acceptable bargains always exist.

$c_i > 0$ , the surplus is always positive. The algebraic expression above simply establishes the existence of the bargaining range by showing that it is possible to give both players their certainty equivalents for war. This calculation further shows you exactly where this range is located, which means that not only is it common knowledge that mutually acceptable deals exist, it is also common knowledge exactly what these deals are.

It should be now obvious that the result also obtains if players are risk-averse because then they would value the certainty equivalent more than the risky war payoff, which means they can be satisfied with shares that are smaller than the certainty equivalents. That is, the bargaining range is larger with these players. It also shows you that the result might not exist if players are risk acceptant. These types of players demand larger certain shares to compensate them for foregoing the risk of the war outcome. Even if the expected war outcomes sum up to less than the benefit, the fact that obtaining certain outcomes yields lower payoffs means that it may well be the case that the sum of the required peace payoffs exceeds the benefit, and so war would be unavoidable.

Fig. 20 (p. 39) includes a status quo distribution of the benefit,  $q \in (0, 1)$ , merely to illustrate that the result is independent of its existence and location. If it is in the bargaining range,  $q \in [w_1, 1 - w_2]$ , then no revision would take place because moving it in either direction would be detrimental to one of the players, and since the other cannot credibly threaten war, there would be no reason to agree to it. If it is not in the bargaining range, as in Fig. 20 (p. 39), where  $q > 1 - w_2$ , then one of the players (in this case, player 2) can credibly threaten war if no revision takes place, and since player 1 prefers any  $x \geq w_1$  to war, there is an incentive to accommodate player 2. The division of the benefit will be revised to something inside the bargaining range, and no war would occur either.

This model establishes the **rationalist puzzle of war**, which goes as follows: Why would players ever fight when mutually acceptable peaceful bargains always exist (and it is common knowledge what they

are) whenever war is costlier than peace?<sup>24</sup>

Observe now that the argument establishes that the bargaining range exists but it does not tell you what deal(s) in it players would coordinate on. This is because the answer to that is dependent on the structure of the negotiation process. For instance, if player 1 can make a take-it-or-leave-it demand, then he will extract the entire bargaining surplus:  $x^* = p + c_2$ , as we have seen happen in the Ultimatum Game. If player 2 were to be given the proposal power, then she would do so:  $x^* = p - c_1$ . In fact, any division  $x \in [p - c_1, p + c_2]$  could be supported with some bargaining protocol. For instance, let the players submit proposals simultaneously, call them  $x_i \in [0, 1]$ , and if they are compatible,  $x_1 + x_2 \leq 1$ , the division is implemented, and if not, then war occurs. Any strategy profile in which player 1 demands  $x_1 > p + c_2$  cannot end peacefully in equilibrium because doing so would yield player 2 less than her war payoff, and she will be strictly better off submitting an incompatible demand instead. Similarly, any strategy profile where player 2 demands  $x_1 < p - c_1$  cannot end peacefully either. All strategy profiles with  $x_1 \in [p - c_1, p + c_2]$  and  $x_2 = 1 - x_1$ , on the other hand, are equilibria. Neither player wants to reduce their demand because doing so would still result in peace but yield a lower payoff. Neither player wants to increase their demand because doing so would lead to war, which cannot improve on their payoffs either.

Thus, the prediction about the precise war-avoiding division depends on the bargaining protocol. Instead of specifying that protocol, let us derive a very general result that will not depend on the protocol. We shall derive a *sufficient condition for war*, which will guarantee that the game cannot end peacefully no matter how players bargain as long as the assumption is maintained that each player's peace payoff must be at least as large as their expected war payoff.

Consider now the same interaction over two periods,  $t \in \{1, 2\}$ , which are structurally identical. If players agree to some distribution of the benefit in  $t = 1$ , then this division is immediately implemented, they receive instantaneous per-period payoffs from it, and the game advances to the second period, where they negotiate *de novo* (that is, irrespective of what the existing agreement is). Agreement ends the game with the appropriate per-period payoffs from the (possibly new) division. The payoff for the game is the sum of the two per-period payoffs.

If players fight in the second period, the outcome is war with payoffs just like in the single-period game. If they fight in the first period, the outcome is settled for both periods, so the expected war payoffs are:

$$\begin{aligned} \text{Player 1 : } & p[u_1(1) + u_1(1)] + (1 - p)[u_1(0) + u_1(0)] - c_1 = 2p - c_1 \\ \text{Player 2 : } & p[u_2(1) + u_2(1)] + (1 - p)[u_2(0) + u_2(0)] - c_2 = 2(1 - p) - c_2. \end{aligned}$$

From our argument above, we know that if players reach the second period, it must always end peacefully. The only possibility for bargaining breakdown and war must be in the first period. With the current specification of the model, however, war will never occur because it is possible to satisfy both sides' war expectations in  $t = 1$  as well. To see this, let  $x_t$  denote period- $t$ 's share for player 1. We know that peace must prevail in the second period with some  $x_2^* \in [w_1, 1 - w_2]$ .

In the first period, player 1 would prefer peace if  $u_1(x_1) + u_1(x_2^*) \geq 2p - c_1$ . This means that he would accept any deal such that  $x_1 \geq 2p - c_1 - x_2^*$ . Using the definition of  $x_2^*$ , the minimum that player 1 would accept is either  $p$  if  $x_2^* = w_1$ , or  $p - (c_1 + c_2)$  if  $x_2^* = 1 - w_2$ . In other words, player 1's minimum demand in the first period depends on what he expects the peaceful deal to be in the second period. Let  $\underline{x}_1 = p$  denote player 1's *largest* minimum demand, and note that it is feasible.

Turning now to player 2, she would prefer peace if  $u_2(x_1) + u_2(x_2^*) \geq 2(1 - p) - c_2$ . This means that she would accept any deal such that  $x_1 \leq 2p + c_2 - x_2^*$ . The bounds of her maximum concession also

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<sup>24</sup>Fearon was the first to state the puzzle in (roughly) these terms in his "Rationalist Explanations for War" in 1995, a seminal piece that redefined research on the causes of war.



depend on  $x_2^*$  and are either  $p + (c_1 + c_2)$  if  $x_2^* = w_1$  or  $p$  if  $x_2^* = 1 - w_2$ . Let  $\bar{x}_1 = p$  denote player 2's *smallest* maximum concession, and observe that it is feasible as well.

Since  $\underline{x}_1 = \bar{x}_2 = p$  — that is, the largest minimum demand that player 1 would ever make does not exceed the smallest maximum concession that player 2 would be willing to make, and this demand is feasible — we conclude that this interaction must end peacefully. There are mutually acceptable deals in the first period as well, and war would never occur in the two-period game either.

Simply making this a dynamic problem does not, in itself, affect the outcome: peace must still prevail as long as the game remains the same over time. One thing that might not remain the same over time, however, is the distribution of power. Let us assume that player 1 starts the game relatively powerful, with a distribution of power  $\hat{p} \in (0, 1)$ , but declines before the second period interaction begins to some  $p \in (0, \hat{p})$ .

This changes nothing in the second period, where the war payoffs are  $w_i$  as before, and so in any SPE, the game must end peacefully there. The new distribution of power, however, does affect the expected war payoffs in the first period, where player 1 can now “lock in” his advantage if he chooses to fight:

$$\hat{w}_1 = 2\hat{p} - c_1 \quad \text{and} \quad \hat{w}_2 = 2(1 - \hat{p}) - c_2.$$

Player 1 would prefer peace if  $u_1(x_1) + u_1(x_2^*) \geq \hat{w}_1$ , which means that he would agree to any  $x_1 \geq 2\hat{p} - c_1 - x_2^*$ . But now we have a problem. The *smallest* minimum demand that player 1 could ever have is when he obtains the best possible deal in the second period:  $x_2^* = p + c_2$ . In this case, peace requires deals such that  $x_1 \geq 2\hat{p} - p - (c_1 + c_2) = \hat{x}_1$ , but this demand might not be feasible. It will be impossible to satisfy player 1 in the first period if  $\hat{x}_1 > 1$ , or if

$$\hat{p} + (\hat{p} - p) > 1 + (c_1 + c_2).$$

This **sufficiency condition** can certainly be satisfied if  $\hat{p}$  is sufficiently large,  $p$  is sufficiently small, and  $c_1 + c_2$  is sufficiently small. (To see this, note that if  $\hat{p}$  is close to 1 and  $p$  is closer to zero, the left-hand side of the inequality is close to 2, and with  $c_1 + c_2$  small enough, the right-hand side will be less than that.) We shall call  $(\hat{p} - p)$  the *size of the power shift*. If the power shift is sufficiently large, then it will not be possible to satisfy player 1's war expectations in the first period and war is inevitable.

The problem is manifested only for the *declining* player 1. To see this note that the *rising* player 2 would prefer peace if  $u_2(x_1) + u_2(x_2^*) \geq \hat{w}_2$ , which means that she would agree to any  $x_1 \leq 2\hat{p} + c_2 - x_2^*$ . This is minimized at  $x_2^* = p + c_2$ , where it reduces to  $x_1 \leq 2\hat{p} - p$ . But since  $2\hat{p} - p > 0$ , it follows that there are always feasible demands that player 2 would agree to. In other words, the problem is that the declining player cannot be compensated enough to accept peace given the disadvantages he expects in the future because of the power shift.

The first insight of the model is that a sufficiently large power shift can lead to a bargaining breakdown (causing war) in the dynamic game. This is so even though if we were to consider each period separately no fighting would occur. The breakdown happens because the two periods are linked strategically with player 1 having the option to impose an expected war outcome for both periods before the power shift occurs. This linkage by itself is not enough if the power shift is not very large. We saw this in the static environment (the sufficiency condition cannot be satisfied if  $\hat{p} = p$ ). More to the point, however, the size of the power shift matters greatly, as the sufficiency condition shows.

Having even a large power shift is, by itself, not enough to trigger the problem. To see this, what feasible demand player 1 could have that would be satisfactory in the first period. Let  $\hat{x}_2$  denote the deal he expects in the second period. Then he would agree to any  $x_1 \geq 2\hat{p} - c_1 - \hat{x}_2$ . Such deals will exist whenever  $2\hat{p} - c_1 - \hat{x}_2 \leq 1$ , or whenever  $\hat{x}_2 \geq 2\hat{p} - c_1 - 1$ . The problem is that player 2 cannot credibly promise to make any such offers when the sufficiency condition holds. To see this, recall that the most she would

concede in the second period is  $p + c_2$ . Thus, if  $2\hat{p} - c_1 - 1 > p + c_2$ , player 1's expectations cannot be met even with the largest concession that she would be willing to make. But this is merely a restatement of the sufficiency condition, so if it is satisfied player 1 cannot be bought off.

This reveals the fundamental problem caused by the power shift. The reason the declining player cannot be compensated for the expected power shift has to do with what the rising player can credibly agree to after the power shift occurs. The key to the bargaining breakdown is player 1's very bad expected peaceful outcome from bargaining in the second period. If player 2 could credibly promise a better deal in that period, then war could be avoided. In particular, if she could credibly commit to offering a 2nd-period deal that would be worse than her expected war payoff in the second period, then it would be possible to satisfy player 1 in the first period. But player 1 knows that when the second period arrives player 2 will have no incentive to abide by that promise: she would be better off rejecting any such deals and fighting. Therefore, the most she can credibly commit to is to relinquish everything over her war payoff,  $x_2^* = 1 - w_2$ . But since the power shift is expected to improve her war payoff substantially, this concession turns out to be too small for player 1 to deter him from fighting on better terms.

This inability of the rising player to credibly promise sufficient future compensation to the declining player is why the bargaining breakdown caused by a large power shift is called the *credible commitment problem*.<sup>25</sup>

## 5 Critiques of Subgame Perfection

Although backward induction and subgame perfection give compelling arguments for reasonable play in simple two-stage games of perfect information, things get uglier once we consider games with many players or games where each player moves several times.

First, consider a game with  $n$  players that has the structure depicted in Fig. 21 (p. 42). Since this is a game of perfect information, we can apply the backward induction algorithm. The unique equilibrium is the profile where each player chooses  $C$  and in the outcome each player gets 2.

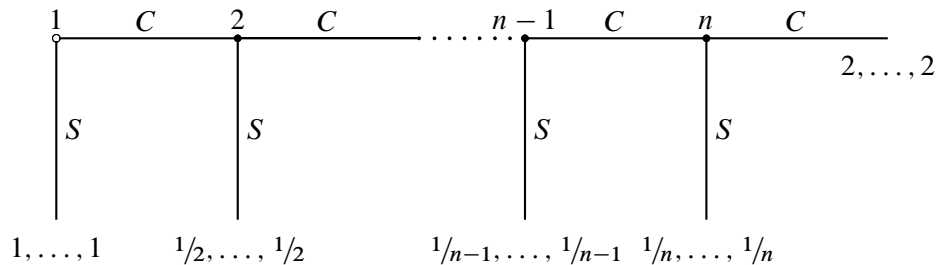


Figure 21: A Game with Many Players.

People have argued that this is unreasonable because in order to get the payoff of 2, all  $n - 1$  players must choose  $C$ . If the probability that any player chooses  $C$  is  $p < 1$ , independent of the others, then the probability that all  $n - 1$  will choose  $C$  is  $p^{n-1}$ , which can be quite small if  $n$  is large even if  $p$  itself is very close to 1. For example, with  $p = .999$  and  $n = 1001$ , this probability is  $(.999)^{1000} \approx .37$ , and with  $n = 10,001$ , it is barely .00005. Moreover, player 1 has to worry that player 2 might have these concerns and might choose  $S$  in order to safeguard either against possible "mistakes" by other players in the future or the possibility that player 3, having these same concerns, might intentionally play  $S$ .

<sup>25</sup>In my formal models in IR class, you will see a more general statement of this result, which also adds the important nuance that the power shift must not only be large but also sufficiently rapid (in the sense that players cannot negotiate interim agreements while it is taking place).

In order for the equilibrium to work, not only must players not make mistakes, but they also must know that everyone else knows the payoffs, and knows that everyone else knows the payoffs, and knows that everyone else knows that everyone else knows the payoffs, and so on and so forth. This is the *common knowledge* assumption that we've seen before. In game theory it is usually assumed that payoffs are common knowledge and so we can use arbitrarily long chains in our solutions. However, some people feel that the longer these chains, the less compelling the solution that requires them.

Another critique is that subgame perfection requires that players agree on the play in a subgame even when backward induction cannot predict the play.

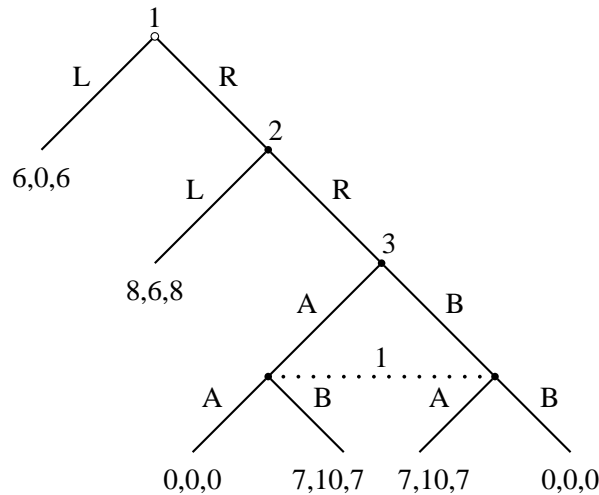


Figure 22: The Coordination Problem Game.

The coordination game between player 1 and 3 has three Nash equilibria: two in pure strategies with payoffs  $(7, 10, 7)$ , and one in mixed strategies with payoffs  $(3.5, 5, 3.5)$ .<sup>26</sup> If we specify an equilibrium in which player 1 and 3 successfully coordinate, then player 2 will choose *R*, and so player 1 will choose *R* as well, expecting a payoff of 7. If we specify the MSNE, then player 2 will choose *L* because *R* yields an expected payoff of 5 (coordination will fail half of the time). Again player 1 will choose *R* expecting a payoff of 8. Thus, in all SPE of this game player 1 chooses *R*.

Suppose, however, player 1 did not see a way to coordinate in the third stage, and hence expected a payoff of 3.5 conditional on this stage being reached, but feared that player 2 would believe that the play in the third stage would result in coordination on an efficient equilibrium. (This is not unreasonable since the two pure strategy Nash equilibria there are the efficient ones.) If player 2 had such expectations, then she would choose *R*, which means that player 1 would go *L* at the initial node!

The problem with SPE is that *all players must expect the same Nash equilibria in all subgames*. So, while this was not a big problem for subgames with unique Nash equilibria, the critique has significant bite in cases like the one just shown. Is such a common expectation reasonable? Who knows? (It depends on the reason the equilibrium arises in the first place, which is not something we can say a whole lot about yet.)

Yet another critique has to do with games where the same player has to move many times. It seems to me that this is a more serious problem. Consider the game depicted in Fig. 23 (p. 44).

The backward-induction solution is that players choose *S* at every information set. However, suppose that contrary to expectations player 1 chooses *C* at the initial node. What should player 2 do? The backward-induction solution says to play *S*, because player 1 will play *S* given a chance. However, player

<sup>26</sup>In this MSNE, each player chooses *A* with probability  $1/2$ , as you should readily see.

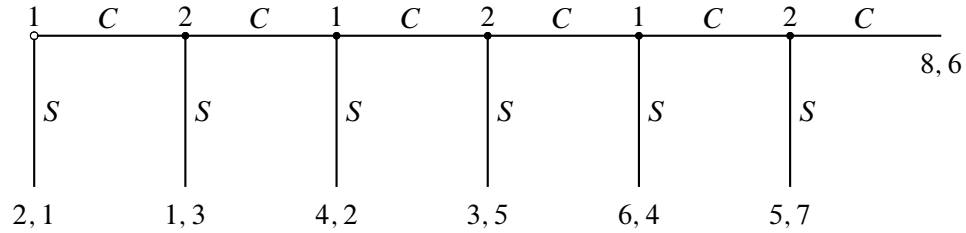


Figure 23: The Centipede Game.

1 should have played  $S$  at the initial node but did not. Since player 2's optimal behavior depends on her beliefs about player 1's behavior in the future, how does she form these beliefs following a 0-probability event? For example, if she believes that player 1 will stop with probability less than  $\frac{2}{3}$ , then she should play  $C$  because doing so will get her at least 3, which is the best she obtains from stopping.

How does player 2 form these beliefs and what beliefs are reasonable? There are two ways to address this problem. First, we may introduce some payoff uncertainty and interpret deviations from expected play by the payoffs differing from those originally thought to be most likely. Instead of conditioning beliefs on probability-0 events, this approach conditions them payoffs that are most likely given the "deviation".

Second, we may interpret the extensive form game as implicitly including the possibility that players sometimes make small "mistakes" or "trembles" whenever they act. If the probabilities of "trembles" are independent across different information sets, then no matter how often past play has failed to conform to the predictions of backward induction, a player is still justified in continuing to use backward induction for the rest of the game. There is a "trembling-hand perfect" equilibrium due to Selten that formalizes this idea. (This is a defense of backward induction.)

The question now becomes one of choosing between two possible interpretations of deviations. In Fig. 23 (p. 44), if player 2 observes  $C$ , will she interpret this as a small "mistake" by player 1 or as a signal that player 1 will choose  $C$  if given a chance? Who knows? I am more inclined toward the latter interpretation but your mileage may vary. To see why it may make sense to treat deviations as a signal, suppose we extend the centipede game to 40 periods and now suppose we find ourselves in period 20; that is, both players have played  $C$  10 times. Is it reasonable to suppose these were all mistakes? Or that perhaps players are trying to get closer to the endgame where they would get better payoffs? In experimental settings, players usually do continue for a while although they do tend to stop well short of the end. One way we can think about this is that the game is not actually capturing everything about the players. In particular, in experiments a player may doubt the rationality of the opponent (so he may expect her to continue) or he may believe she doubts his own rationality (so she expects him to continue, which in turn makes him expect her to continue as well). At any rate, small doubts like this may move the play beyond the game-stopping first choice by player 1. This does not mean that backward induction is "wrong." What it does mean is that the full information common knowledge assumptions behind it may not be captured in experiments where real people play the Centipede Game. My reaction to this is not to abandon backward induction but to modify the model and ask: what will happen if players with small doubts about each other's rationality play the Centipede Game? This is a topic for another discussion, though.